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On the Geometry of Numbers of Non-Convex Star-Regions with Hexagonal Symmetry

R. P. Bambah

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ON THE GEOMETRY OF NUMBERS OF NON-CONVEX STAR-REGIONS WITH HEXAGONAL SYMMETRY

By R. P. BAMBAH, Ph.D., *St John's College, University of Cambridge**

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CONTENTS

A method of Mordell is applied to the study of critical determinants and critical lattices of a type of two-dimensional star-domains which can be roughly described as that of regions similar to $\lvert r^3 \sin 3\theta \rvert \leqslant 8c^3$. In part I three general theorems are proved. They are applied in part II to obtain the critical determinants and all the critical lattices of some special regions of this type.

PART I. GENERAL THEOREMS

1. Introduction

The geometry of numbers for non-convex regions has in recent years attracted great attention, and many special regions have been studied by various authors. General results have also been obtained by Mordell (1945, 1946) and Mahler (1946*a*, *b*).

Mordell (1945) considered the type of star-regions R defined by the inequality

$$\lvert f(\lvert x \rvert, \lvert y \rvert) \rvert \leqslant c \lvert f(1,1) \rvert,$$

where, for $x \geqslant 0$, $y \geqslant 0$, (i) $f(x,y)$ is defined, is symmetrical in x and y and is homogeneous (of dimension 1, say), (ii) the region $\lvert f(x,y) \rvert \geqslant \lvert f(1,1) \rvert$ is convex, and (iii) the boundary of

* Now at University College, London.

R either terminates on the axes or has them as asymptotes. In other words the regions R have a symmetry which can be called rectangular.

Before stating Mordell's results it will be convenient to give a few standard definitions.

(1) Let R be any region and $\mathscr{L}$ a lattice with no point except O in the interior of R. Then $\mathscr{L}$ is said to be *admissible for R* or simply *R-admissible*.

(2) The lower bound of the determinants of all R-admissible lattices is called the *critical determinant of R* and is usually denoted by $\Delta(R)$.

(3) R-admissible lattices of determinant $\Delta(R)$ are called *critical lattices of R*.

For his regions R Mordell showed that there exist a suitable number Δ and two suitable points $A(a, b)$ and $B(b, a)$ on the first quadrant boundary of R such that we have the following theorem.

THEOREM. *Let P be any point lying between A and B on the boundary of R. Then a line $Q_1 Q_2$ equal and parallel to OP can be drawn such that Q_1 and Q_2 lie on the second quadrant boundary of R. If, for all such points P, the area of the parallelogram $OPQ_1 Q_2$ is not less than Δ, then every lattice of determinant Δ or less has a point other than O in the closed region R, i.e. $\Delta \leqslant \Delta(R)$.*

Further, if for some P the area of $OPQ_1 Q_2$ is equal to Δ and if the corresponding lattice $\mathscr{L}$ generated by P and Q_1 is R-admissible, then $\Delta = \Delta(R)$ and $\mathscr{L}$ is a critical lattice of R.†

To establish the property postulated for R it is necessary to solve an extremal problem, which, though simple in theory, may be difficult in practice. Mordell showed that the problem can be solved for many particular regions.

Let $l_4 Ol_1$, $l_5 Ol_2$ and $l_6 Ol_3$ be three lines through O such that Ol_2 and Ol_3 make angles of 120° and 240° respectively with Ol_1. These lines divide the plane into six congruent parts. At Professor Mordell's suggestion I have studied the possibility of extending his method to the case of similar star-regions R, symmetrical with respect to these lines and their bisectors and satisfying suitable convexity conditions. Obviously every such region R consists of six parts, all congruent. Such an R will be called a *region with hexagonal symmetry*.

It proves to be convenient to divide these regions into four types. For three of them, without postulating any extremal conditions for R, a number $\Delta \leqslant \Delta(R)$ is obtained. In one of these cases it is shown that there is just one lattice of determinant Δ admissible for R so that $\Delta = \Delta(R)$ and there is just one critical lattice of R. In each of the other two cases there exist just two lattices of determinant Δ, the admissibility of which is the necessary and sufficient condition for Δ to be equal to $\Delta(R)$.

For R of the fourth type the problem is reduced to an extremal one of Mordell's type. First, a suitable number Δ and two suitable points H_1 and I_1 are found with H_1 and I_1 lying on the boundary of R between Ol_1 and Ol_6. (The points H_1 and I_1 lie between two other points A_1 and B_1 on the boundary of R. The points A_1 and B_1 have importance for critical lattices and are also easier to employ in applications.) Then the following theorem is proved:

Let P be a point lying on the boundary of R between H_1 and I_1, or coincident with A_1 or B_1. Lines $P_1 P_2$ and $P_3 P_4$ can be drawn equal and parallel to OP with the points P_1, P_2 lying on the second sector boundary of R and P_3, P_4 on the third. Suppose that for all such P one at least of the parallelograms $OPP_1 P_2$, $OPP_3 P_4$ has an area not less than Δ, then $\Delta \leqslant \Delta(R)$.

† In particular, the admissibility of lattices corresponding to A and B is a sufficient condition for the equality of Δ and $\Delta(R)$. For bounded R, Mordell gave sufficient conditions for the admissibility of these lattices. He could prove the admissibility of these lattices for some unbounded regions also.

If for some P the larger of the parallelograms OPP_1P_2 *and* OPP_3P_4 *has area equal to* Δ *and the corresponding lattice* *$\mathscr{L}$ *is admissible, then* $\Delta = \Delta(R)$ *and* $\mathscr{L}$ *is a critical lattice of R.*

It is interesting to note that *for three of the four types it has been possible to avoid the extremal condition* which is necessary for the fourth type only.

In part II of the paper the results of part I are applied to obtain the critical determinants and all the critical lattices for the following regions.

(1) $|r^3 \sin 3\theta| \leqslant 8c^3$. The result for this region is equivalent to Mordell's famous theorem on the minimum of a binary cubic form with a positive discriminant.

(2) Regions with hexagonal symmetry bounded by circular arcs.

(3) Regions with hexagonal symmetry bounded by parabolic arcs.

(4) Regions with hexagonal symmetry bounded by hyperbolic arcs with asymptotic angles $2\theta \geqslant \pi - 2\tan^{-1}\surd(159/57)$.

(5) Twelve-sided stars with vertical angles 2θ, where $\tan^{-1} 0{\cdot}4221/\surd 3 \leqslant \theta \leqslant \frac{1}{2}\pi$.

2. Description and analytical formulation of some properties of R

The assumptions about the star-region R are as follows. It is symmetrical with respect to the lines l_1Ol_4, l_2Ol_5, l_3Ol_6 and their bisectors. Its boundary $\mathscr{C}$ either terminates on the lines l_1Ol_4, l_2Ol_5 and l_3Ol_6 or has them as asymptotes. The region $\mathscr{T}$, external to R and lying between Ol_1 and Ol_6, is convex. Each of the six parts of $\mathscr{C}$ is a continuous curve.

The analytical equivalents of these assumptions and of certain properties of R which can be derived from them are given below. Regarding the distance of a point P from a directed line l as positive or negative according as P is on the left or right of l, take the x, y and z co-ordinates of a point to be its distances (with the proper sign) from the directed lines l_4Ol_1, l_5Ol_2 and l_6Ol_3 respectively. (Obviously for every P, $x+y+z=0$.)

Then R can be defined by the inequality

$$f(x,y,z) \leqslant f(c,-2c,c),$$

where f is a non-negative function of x, y, z.

As R is a star-region bounded by continuous curves, we have

$$\left.\begin{array}{c}\textit{for } x\geqslant 0, z\geqslant 0, f(x,y,z) \textit{ is continuous;}\\ f(0,0,0)=0; \textit{ and for } t\geqslant 0, f(tx,ty,tz)=tf(x,y,z).\end{array}\right\} \tag{2·1}$$

The symmetry conditions imply symmetry about the origin. Therefore

$$f(-x,-y,-z)=f(x,y,z). \tag{2·2}$$

After (2·2) the symmetry conditions are easily seen to be equivalent to the invariance of R under a rotation of 120° or a reflexion in the line $x=z$. From this fact it easily follows that

$$f(x,y,z) \quad \textit{is symmetric in all the variables.} \tag{2·3}$$

It is obviously enough for a full description of R to describe its part in the sector $x\geqslant 0, z\geqslant 0$. It will therefore be supposed in the rest of the section that $x\geqslant 0, z\geqslant 0$.

The convexity condition is equivalent to

$$f(x_1+x_2, y_1+y_2, z_1+z_2) \geqslant f(x_1,y_1,z_1)+f(x_2,y_2,z_2). \tag{2·4}$$

* $\mathscr{L}$ is generated by P and P_1 or P and P_3 according as OPP_1P_2 or OPP_3P_4 has area equal to Δ.

58-2

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

Let $P_1(x_1, y_1, z_1)$ be any fixed point. Consider the region $\mathscr{T}'$ defined by the inequality $f(x, y, z) \geqslant f(x_1, y_1, z_1)$. The curve $f(x, y, z) = f(x_1, y_1, z_1)$ either has the line $x = 0$ as an asymptote or terminates on it. Therefore, the point at ∞ on the line $x = x_1$ lies in $\mathscr{T}'$, so that, by the convexity of $\mathscr{T}'$, any point on the line $x = x_1$ to the right of P lies in $\mathscr{T}'$. In other words, if $z_2 > z_1$ then $f(x_1, y_2, z_2) \geqslant f(x_1, y_1, z_1)$. Therefore,

$$\textit{for fixed } x, f(x, y, z) \textit{ is a non-decreasing function of } z. \tag{2·5}$$

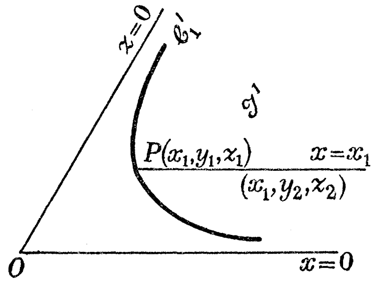

FIGURE 1

Similarly, using the fact that the region either terminates on $z = 0$ or has it as an asymptote, it can be shown that

$$\textit{for fixed } z, f(x, y, z) \textit{ is a non-decreasing function of } x. \tag{2·6}$$

Let P_1 be the point $(1-r, -2, 1+r)$, where $0 \leqslant r \leqslant 1$. As $0 \leqslant 1-r \leqslant 1+r$, P_1 lies in the lower half of the sector $x \geqslant 0, z \geqslant 0$. Consider the region $\mathscr{T}'$ defined by the relation

$$f(x, y, z) \geqslant f(1-r, -2, 1+r).$$

Let its boundary be denoted by $\mathscr{C}'$. Suppose the line $\Lambda_2 : y = -2$ is not a part of $\mathscr{C}'$. As the region $\mathscr{T}'$ is convex and symmetrical about the line $\Lambda_1 : x = z$ and Λ_2 is perpendicular to Λ_1, the only points in which Λ_2 meets $\mathscr{C}'$ are P_1 and P_2, the image of P_1 in Λ_1. Consequently every point on Λ_2 lying outside the segment P_1P_2 lies outside $\mathscr{T}'$. As P_1 is below Λ_1, P_2 is above it, so that every point on Λ_2, which lies between $x = 0$ and P_1, lies outside $\mathscr{T}'$. This means that if $0 \leqslant r_2 \leqslant 1$ and $r_2 > r$, then $f(1-r_2, -2, 1+r_2) < f(1-r, -2, 1+r)$. In case Λ_2 forms a part of $\mathscr{C}'$, any point on it either lies on $\mathscr{C}'$ or outside $\mathscr{T}'$, so that if $0 \leqslant r_2 \leqslant 1$, then $f(1-r_2, -2, 1+r_2) \leqslant f(1-r, -2, 1+r)$. From all this it follows that

$$\textit{if } 0 \leqslant r \leqslant 1, \textit{ then } f(1-r, -2, 1+r) \textit{ is a non-increasing function of } r. \tag{2·7}$$

Let P be the point with co-ordinates $(a-r, -2a+\frac{1}{2}r, a+\frac{1}{2}r)$ and $\mathscr{T}''$ the region

$$f(x, y, z) \geqslant f(a-r, -2a+\tfrac{1}{2}r, a+\tfrac{1}{2}r).$$

Suppose the co-ordinates of P', the point where $\Lambda_1 : x = z$, meets $\mathscr{C}''$, the boundary of $\mathscr{T}''$, are given by $(c_1, -2c_1, c_1)$. Then, because of its convexity and symmetry about Λ_1, $\mathscr{T}''$ has Λ_3, the line through P' perpendicular to Λ_1, as a tangent or a tac-line. Consequently the region $\mathscr{T}''$, and in particular the point P, lies to the right of $\Lambda_3 : y + 2c_1 = 0$. Therefore, $a > c_1$, and $(a, -2a, a)$ the point, where $z - y = 3a$ meets Λ_1 lies strictly inside $\mathscr{T}''$. From this it follows that any point on $z - y = 3a$ lying below P is strictly outside $\mathscr{T}''$. This means that if $r_1 > r$ and $0 \leqslant a - r_1 \leqslant a + \frac{1}{2}r_1$, then $f(a-r_1, -2a+\frac{1}{2}r_1, a+\frac{1}{2}r_1) < f(a-r, -2a+\frac{1}{2}r, a+\frac{1}{2}r)$. In other words

$$\textit{for } 0 \leqslant a-r \leqslant a+\tfrac{1}{2}r, f(a-r, -2a+\tfrac{1}{2}r, a+\tfrac{1}{2}r) \textit{ is a strictly decreasing function of } r. \tag{2·8}$$

It may be observed in conclusion that if R is bounded, then $f(x, y, z) = 0$ only for $(x, y, z) = (0, 0, 0)$. On the other hand, if R is unbounded, then

$$f(0, a, -a) = f(a, 0, -a) = f(a, -a, 0) = 0$$

for all a.

It will be convenient to call the sectors into which the plane is divided by the lines $x = 0$, $y = 0$ and $z = 0$, sectors I, II, III, IV, V and VI, starting with the sector $x \geqslant 0$, $z \geqslant 0$ and moving in the counter-clockwise direction. Also, if a point is denoted by P_1, then P_2, P_3, P_4, P_5 and P_6 will denote the points obtained from it by the rotations through 60°, 120°, 180°, 240° and 300° respectively.

3. The first type of regions R

Let R be defined by $f(x, y, z) \leqslant f(c, -2c, c)$, $x+y+z = 0$. R will be said to belong to the first type if, in addition to the conditions (2·1) to (2·8), $f(x, y, z)$ satisfies the further condition

$$f(0, -3c, 3c) \geqslant f(c, -2c, c). \tag{3·1}$$

THEOREM 1. *If $R : f(x, y, z) \leqslant f(c, -2c, c)$, is a region of the first type, then $\Delta(R) = 2c^2\sqrt{3}$, and the only critical lattice is the one generated by the mid-points of its boundary arcs.*

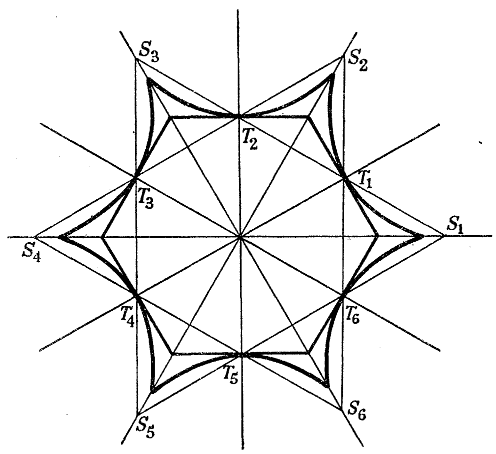

FIGURE 2

Proof. Let the point $(c, -2c, c)$ be called T_1 (see figure 2). From the symmetry and convexity conditions it can be easily seen that the straight lines perpendicular to lines OT_i at points T_i $(i = 1, 2, 3, \ldots, 6)$ enclose a regular hexagon H, centred at O and lying in R. The area of H is given by $6OT_1^2 \tan \frac{1}{6}\pi = 8c^2\sqrt{3}$. As H is a subset of R, it is clear that

$$2c^2\sqrt{3} = \Delta(H) \leqslant \Delta(R).$$

Further, it is known that H has just one critical lattice, namely, $\mathscr{L}$, the lattice generated by points T_i. It will suffice, therefore, to show that this lattice is R-admissible.

Let S_1 denote the point $T_6 + T_1$ in vector notation. Its co-ordinates are $(0, -3c, 3c)$. Therefore by (3·1) S_1 is external to R and so are $S_2, S_3, \ldots, S_6$. Here the exterior of R is supposed to include its boundary. By the convexity condition, R lies entirely within the hexagon $S_1S_2S_3S_4S_5S_6$. The only points of $\mathscr{L}$ within this hexagon are $T_1, T_2, \ldots, T_6$, all of which lie on the boundary of R. Therefore, $\mathscr{L}$ is R-admissible and the theorem is completely proved.

4. Determination of the points A_1, B_1 and the number Δ

In the rest of part I it will be supposed that $f(x,y,z)$ satisfies, in addition to (2·1) to (2·8), the further condition

$$f(0,-3,3)<f(1,-2,1). \tag{4·1}$$

It will now be shown that in the half-sector $0\leqslant x\leqslant z$ there exists a unique point A_1 such that the points A_1, A_2 and A_1+A_2 lie on $\mathscr{C}$, the boundary of R. This point A_1 and its image B_1 in the line $x=z$ are the two points referred to in the introduction. Denote A_1+A_2 by C_1 and its image in $x=z$ by D_1. Then from the symmetry of R it is obvious that all points $A_i, B_i, C_i, D_i\,(i=1,2,\ldots,6)$ lie on $\mathscr{C}$.

Now let the co-ordinates of A_1 be $(a,-a-b,b)$. Then those of A_2 are $(a+b,-b,-a)$ and the existence and uniqueness of A_1 follow from

Lemma 1. *There exist unique non-negative numbers a and b, not both zero, such that*

$$f(c,-2c,c)=f(a,-a-b,b)=f(2a+b,-a-2b,b-a). \tag{4·2}$$

Further,

$$0<\tfrac{1}{2}b\leqslant a<b. \tag{4·3}$$

Proof. By the symmetry of $f(x,y,z)$, the relations (4·2) are equivalent to

$$f(c,-2c,c)=f(a,-a-b,b), \tag{4·4}$$

and

$$\begin{aligned}0&=f(a,-a-b,b)-f(2a+b,-a-2b,b-a)\\&=f(a,-a-b,b)-f(b-a,-a-2b,2a+b)\\&=(a+b)\{\tfrac{1}{2}f(1-r,-2,1+r)-f(1-r_1,-2+\tfrac{1}{2}r_1,1+\tfrac{1}{2}r_1)\},\end{aligned} \tag{4·5}$$

where

$$r=(b-a)/(b+a) \quad \text{and} \quad r_1=2a/(a+b).$$

As $a+b\neq 0$, (4·5) is equivalent to

$$\begin{aligned}0&=\tfrac{1}{2}f(1-r,-2,1+r)-f(1-r_1,-2+\tfrac{1}{2}r_1,1+\tfrac{1}{2}r_1)\\&=g_1(a/b)+g_2(a/b) \quad \text{(say)}\\&=g(a/b).\end{aligned} \tag{4·6}$$

Now, as a/b increases from 0 to 1, r decreases from 1 to 0, while r_1 increases from 0 to 1. Therefore, for $0\leqslant a/b\leqslant 1$, it follows from (2·7) and (2·8) respectively that $g_1(a/b)$ and $g_2(a/b)$ are respectively increasing and strictly increasing functions of a/b. This implies that $g(a/b)$ is a strictly increasing function of a/b for $0\leqslant a/b\leqslant 1$. Now, since

$$g(\tfrac{1}{2})=f(\tfrac{1}{3},-1,\tfrac{2}{3})-f(\tfrac{1}{3},-\tfrac{5}{3},\tfrac{4}{3})\leqslant 0 \quad \text{(by (2·5))},$$

and

$$g(1)=f(\tfrac{1}{2},-1,\tfrac{1}{2})-f(0,-\tfrac{3}{2},\tfrac{3}{2})>0 \quad \text{(by (4·1))},$$

the lemma follows at once.

Lemma 1·1. $a^2+ab+b^2\geqslant 3c^2$.

Proof. Let T_1 denote the point $(c,-2c,c)$. Then, by the symmetry and convexity conditions $OA_1\geqslant OT_1$. Therefore

$$\tfrac{4}{3}(a^2+ab+b^2)\geqslant 4c^2,$$

and the lemma follows.

Definition of Δ. Define the number Δ to be the area of the parallelogram $OA_1C_1A_2$, i.e.

$$\Delta=\frac{2}{\sqrt{3}}(a^2+ab+b^2). \tag{4·7}$$

5. Description of figure 3

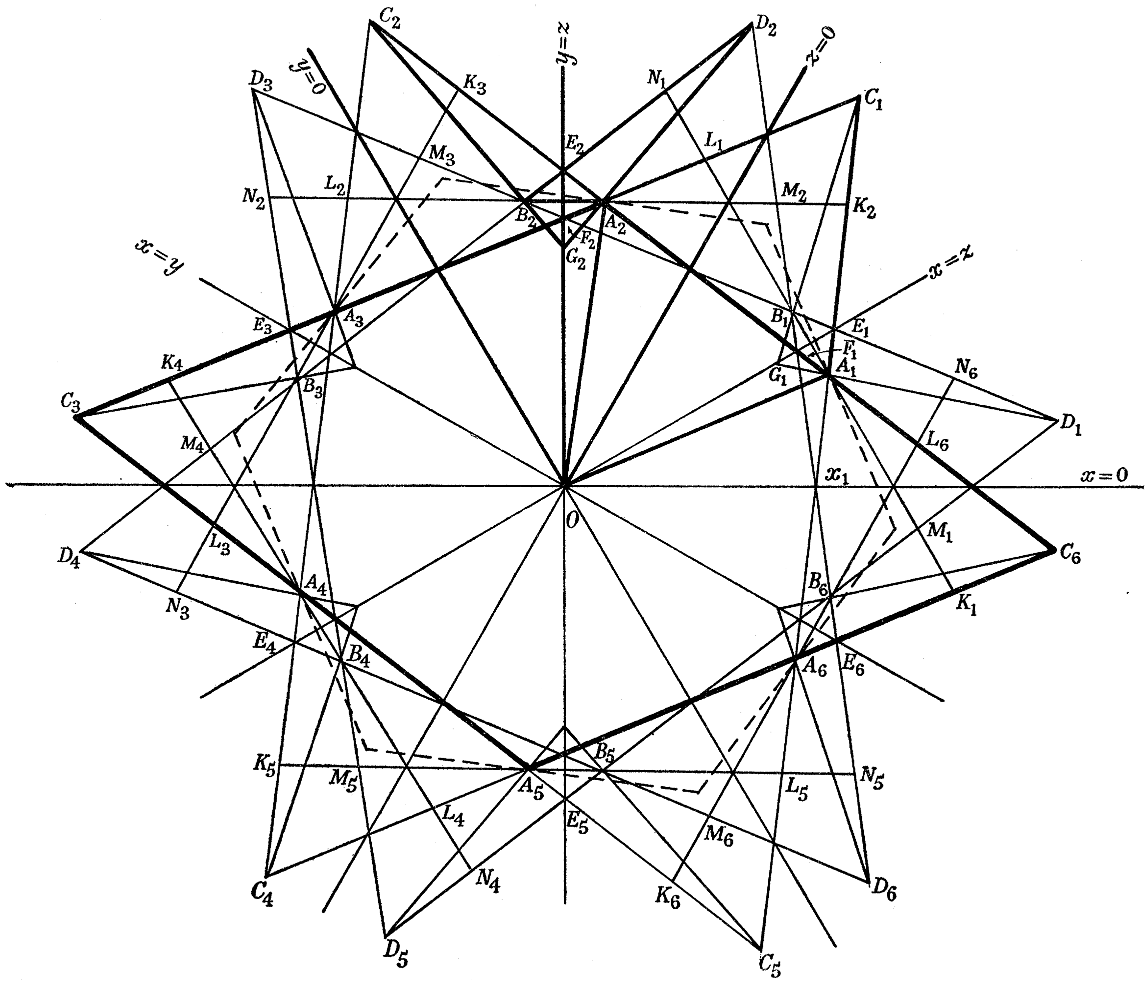

Figure 3

Table 1 gives the co-ordinates of the points A_i, B_i, C_i and D_i $(i = 1, 2, 3)$.

Table 1

A_1:	$(a, -a-b, b)$	A_2:	$(a+b, -b, -a)$	A_3:	$(b, a, -a-b)$
B_1:	$(b, -a-b, a)$	B_2:	$(a+b, -a, -b)$	B_3:	$(a, b, -a-b)$
C_1:	$(2a+b, -a-2b, b-a)$	C_2:	$(a+2b, a-b, -b-2a)$	C_3:	$(b-a, 2a+b, -a-2b)$
D_1:	$(b-a, -a-2b, 2a+b)$	D_2:	$(a+2b, -b-2a, a-b)$	D_3:	$(2a+b, b-a, -a-2b)$

Now follow the equations of a few lines:

$$A_1A_2:\quad ax-by = a^2+ab+b^2; \qquad B_1B_2:\quad bx-ay = a^2+ab+b^2;$$
$$A_2A_3:\quad bx-az = a^2+ab+b^2; \qquad B_2B_3:\quad ax-bz = a^2+ab+b^2;$$
$$A_2B_2:\quad x = a+b; \qquad A_2D_2:\quad 2ax+by = 2a^2+2ab-b^2.$$

Define the points E_2, F_2 and G_2 as follows:

E_2: the intersection of A_2C_2 and B_2D_2.
F_2: the intersection of C_1A_2 and D_3B_2.
G_2: the intersection of D_2A_2 and C_2B_2.

Then, the co-ordinates of E_2, F_2 and G_2 are

$$E_2: \left(\frac{2(a^2+ab+b^2)}{(b+2a)}, \quad -\frac{(a^2+ab+b^2)}{(b+2a)}, \quad -\frac{(a^2+ab+b^2)}{(b+2a)}\right),$$

$$F_2: \left(\frac{2(a^2+ab+b^2)}{(a+2b)}, \quad -\frac{(a^2+ab+b^2)}{(a+2b)}, \quad -\frac{(a^2+ab+b^2)}{(a+2b)}\right),$$

$$G_2: \left(\frac{2(2a^2+2ab-b^2)}{(4a-b)}, \quad -\frac{(2a^2+2ab-b^2)}{(4a-b)}, \quad -\frac{(2a^2+2ab-b^2)}{(4a-b)}\right).$$

The relative positions of points and lines in figure 3

Because of the symmetry it is sufficient to consider only sector II. The statement 'AB is above CD', will mean that the line segment AB is above the infinite line obtained by producing CD in both directions.

D_2B_2 lies above A_2B_2, since D_2 is above A_2B_2, for $a+2b>a+b$.

F_2 lies below A_2B_2, since $\dfrac{2(a^2+ab+b^2)}{(a+2b)}<a+b$, because $a^2<ab$.

G_2 is below F_2, since

$$\frac{2a^2+2ab-b^2}{4a-b}<\frac{a^2+ba+b^2}{a+2b},$$

i.e. $$2a^3+6a^2b+3ab^2-2b^3<4a^3+3a^2b+3ab^2-b^3$$

follows from $$2a^3+b^3-3a^2b=(b-a)^2(b+2a)>0.$$

If (x_1, y_1, z_1) are the co-ordinates of a point P, write

$$x_1 = P(x), \quad y_1 = P(y), \quad z_1 = P(z), \quad z_1-y_1 = P(z-y), \text{ etc.}$$

Then C_1 lies to the right of D_2, which lies to the right of A_2, since

$$C_1(z-y)>D_2(z-y)>A_2(z-y),$$

because $$3b>3a>b-a,$$

$$D_2(x)>C_1(x), \quad \text{since} \quad b>a.$$

$$D_1(x)\leqslant A_1(x), \quad \text{since} \quad b\leqslant 2a.$$

Next observe that the line through A_2 perpendicular to OA_2 lies between A_2B_2 and A_2C_2, for
(i) $\angle B_2A_2O<\frac{1}{2}\pi$, since B_2A_2 is parallel to $x=0$, and
(ii) $\angle C_2A_2O=\angle C_1A_1O>\angle C_1x_1O$ (where x_1 is the point where C_1A_1 meets $x=0$)

$$>\tfrac{1}{2}\pi,$$

since the slope of $A_1C_1=\surd(3)\,(a+b)/(b-a)>0$.

From the above it follows that *the lines perpendicular to OA_i at A_i ($i=1, 2, \ldots, 6$) form a hexagon whose parts, that do not lie in R, lie entirely inside the segments $A_1B_1C_1, A_2B_2C_2, \ldots, A_6B_6C_6$.* By a segment $A_iB_iC_i$ is meant the closed region bounded by the chord A_iC_i and the arc $A_iB_iC_i$ of $\mathscr{C}$.

Since the triangles OA_1A_2, OA_2A_3 and $A_2C_2A_3$ are equilateral, it is easily seen that A_1A_2 is equal to A_2C_2, is parallel to OA_3 and passes through C_2. Similarly, A_1A_2 passes through C_6 and C_2, A_2A_3 through C_1 and C_3, etc. $A_1A_2=OA_3=A_2C_2=C_6A_1$, etc., OA_1 is parallel to A_2A_3, OB_1 to B_2B_3 and so on. A_1B_1, A_2B_2 and A_3B_3 are parallel respectively to $y=0$, $x=0$ and $z=0$.

It is now obvious that each of the parallelograms

$$C_6A_2C_3A_5,\quad C_1A_3C_4A_6,\quad C_2A_4C_5A_1,\quad D_1B_2D_4B_5,\quad D_2B_3D_5B_6\quad \text{and}\quad D_3B_4D_6B_1$$

has its area equal to 4Δ.

6. The types II, III and IV

Let the straight line C_1A_2 meet $\mathscr{C}_2$, the second sector boundary of R, at the point H_2. The regions are classified into types II, III and IV by consideration of the relative positions of the points A_2, H_2 and F_2 as follows. R is said to be of type

II: if A_2 lies between H_2 and F_2 (or if A_2 coincides with H_2) (see figure 4);
III: if H_2 lies between A_2 and F_2 (figure 5);
IV: if F_2 lies between A_2 and H_2 (figure 6).

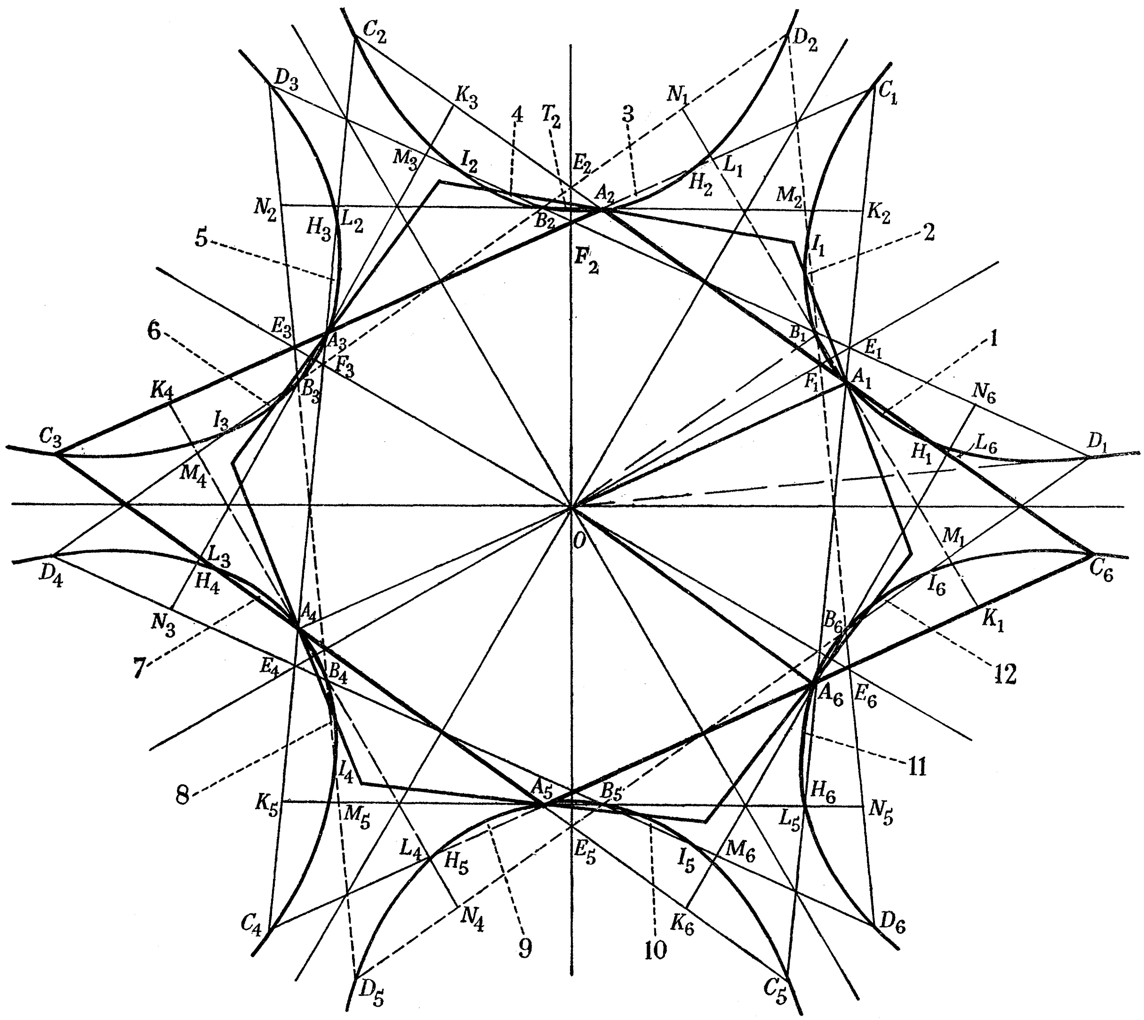

Figure 4

If a part of C_1A_2 to the left of A_2 forms a part of $\mathscr{C}$, R is included in type III. If $\mathscr{C}$ contains a segment of the line C_1A_2, extending on both sides of A_2, R can be taken to belong to type II, when considering the segment on the right of A_2, while R can be taken to be of type III when considering the segment on the left of A_2. As theorem 2 applies to both types II and III, it will cover this type of R, too.

Note that the point $T_2(2c, -c, -c)$ lies above the line C_1A_2 in cases II and III but below it in case IV. The case when T_2 lies on A_2C_1, i.e. F_2 and H_2 coincide, or A_2F_2 forms a part of $\mathscr{C}_2$, may be included in type III.

The reflexion of H_2 in the line $z = y$ is denoted by I_2.

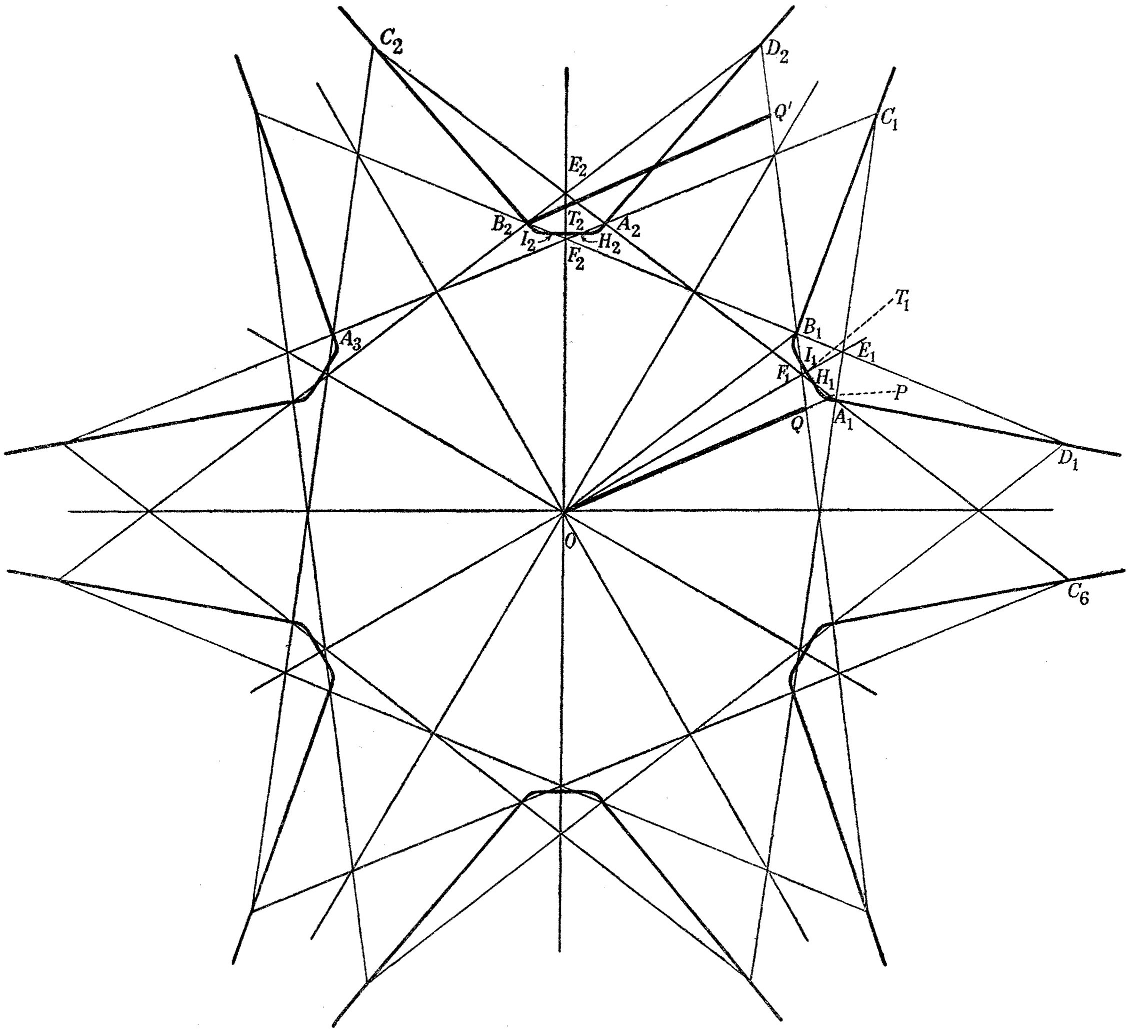

FIGURE 5

7. STATEMENT OF MAIN THEOREMS

Let the lattices generated by A_1 and A_2 and by B_1 and B_2 be denoted by $\mathscr{L}_1$ and $\mathscr{L}_2$.

THEOREM 2. *For R of type II or III, $\Delta(R)$ is not less than Δ. Also, $\Delta(R)$ is equal to Δ if and only if $\mathscr{L}_1$ and $\mathscr{L}_2$ are R-admissible.*

THEOREM 3. *Let R belong to type IV. Let P be either A_1 or B_1, or a point on $\mathscr{C}_1$ between H_1 and I_1, inclusive. Straight lines P_1P_2 and P_3P_4 can be drawn equal and parallel to OP with points P_1 and P_2 lying on $\mathscr{C}_2$, and P_3 and P_4 on $\mathscr{C}_3$, where $\mathscr{C}_2$ and $\mathscr{C}_3$ denote the parts of $\mathscr{C}$ in the second and third sectors respectively. Suppose, for all these P, the area of one at least of the parallelograms OPP_1P_2, OPP_3P_4 is not less than Δ. Then $\Delta(R) \geqslant \Delta$.*

If, for some P, the area of the greater of the parallelograms OPP_1P_2, OPP_3P_4 is equal to Δ and the corresponding lattice **$\mathscr{L}'$ is admissible, then $\Delta(R) = \Delta$ and $\mathscr{L}'$ is a critical lattice of R. In particular $\mathscr{L}_1$ and $\mathscr{L}_2$ are critical if admissible.*

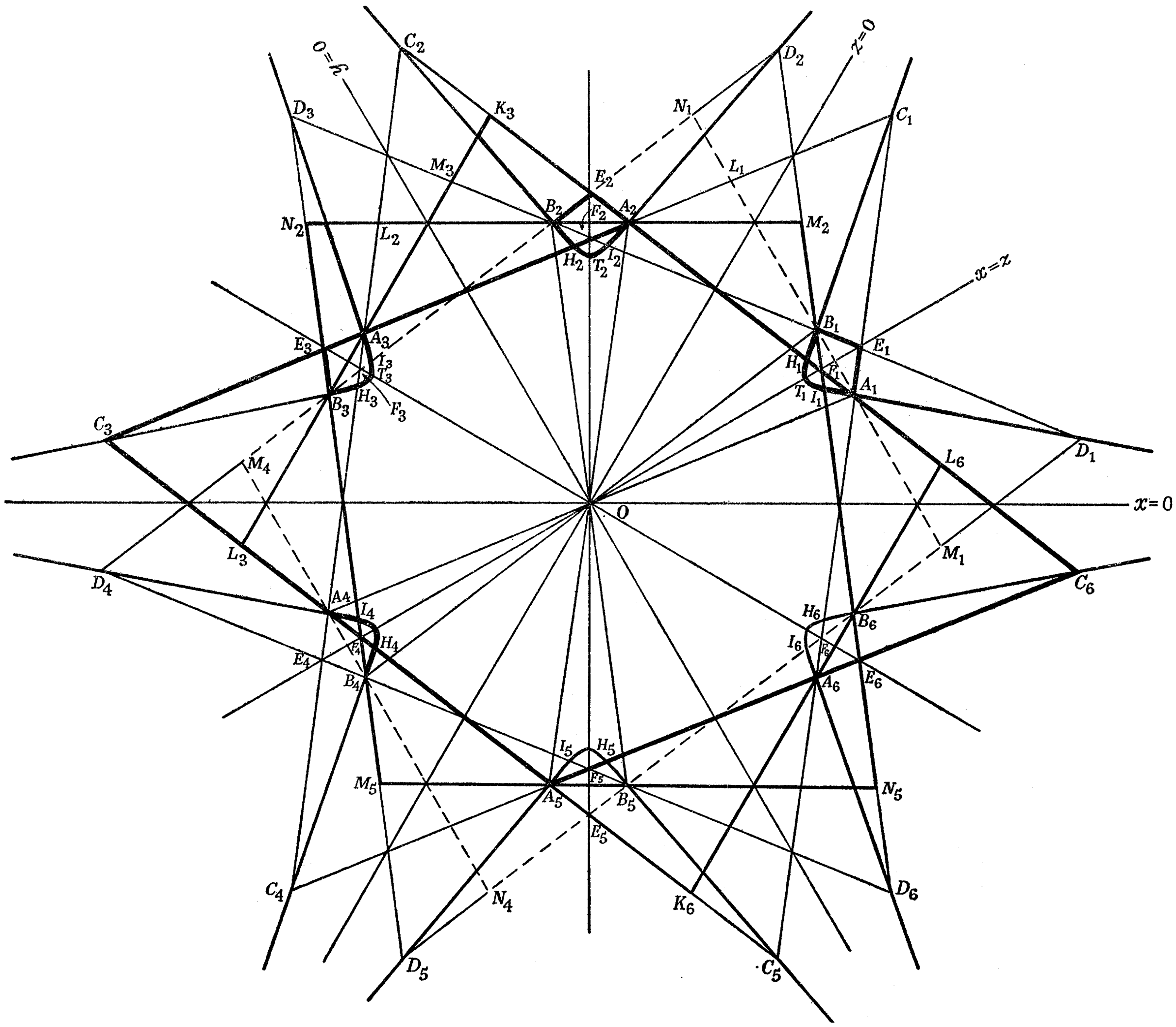

FIGURE 6

8. SOME SIMPLE LEMMAS

A few well-known lemmas which will be employed in the proofs of the above theorems are given below.

LEMMA 2 (Minkowski). *Let R be a closed convex region symmetrical with respect to O and having area equal to 4Δ. Then every lattice $\mathscr{L}$ of determinant not greater than Δ has at least two points inside or on the boundary of R. These points may lie only on the boundary of R only if the determinant of $\mathscr{L}$ is equal to Δ.*

LEMMA 3. *Let π be a parallelogram with a vertex at O and with its area equal to Δ. Then, if $\mathscr{L}$ is a lattice of determinant Δ, π cannot contain two points P and Q of $\mathscr{L}$, non-collinear with O, unless one of them, P say, coincides with a vertex on a side of π through O and the other Q lies on the side parallel to OP.*

* $\mathscr{L}'$ is generated by P and P_1 or P and P_3 according as the area of OPP_1P_2 or OPP_3P_4 is equal to Δ.

Lemma 4. *If $\mathscr{L}$ is a lattice of determinant Δ and OPQ, a triangle with area $\frac{1}{2}\Delta$, then OPQ cannot contain two points of $\mathscr{L}$, non-collinear with O, unless they coincide with P and Q.*

Lemma 5. *Suppose $\mathscr{L}$ is a lattice of determinant Δ and P is a primitive lattice point of $\mathscr{L}$, i.e. the line segment OP contains no points of $\mathscr{L}$ other than O and P. Let Λ be a line parallel to OP and at a distance Δ/OP from it. Then any segment of Λ of length OP contains one and only one lattice point, except that it contains two if the end-points belong to $\mathscr{L}$. $\mathscr{L}$ can be generated by P and any lattice point on Λ.*

9. Outline of the proof

Suppose that there exists an R-admissible lattice $\mathscr{L}'$ of determinant Δ different from $\mathscr{L}_1$ and $\mathscr{L}_2$. It is first shown with the help of certain parallelograms of area 4Δ that $\mathscr{L}'$ has a point P in one at least of $[A_1, E_1, B_1]$, $[A_2, E_2, B_2]$ and $[A_3, E_3, B_3]$, where $[A_i, E_i, B_i]$ denotes the closed region bounded by the lines A_iE_i and E_iB_i and the part of $\mathscr{C}$ lying between A_i and B_i; the points A_i and B_i, however, are supposed to be excluded from $[A_i, E_i, B_i]$. Without loss of generality it may be assumed that P lies in $[A_1, E_1, B_1]$. Then it is shown that neither $[A_2, E_2, B_2]$ nor $[A_3, E_3, B_3]$ contains a point of $\mathscr{L}$. For R of type II this is shown to lead to a contradiction which proves the theorem in this case. For R of type III or IV, it is proved that P must lie in $[A_1, F_1, B_1]$ (see figure 6 for example).

It is next shown that, if a line parallel to OP and at a distance Δ/OP from it has an intercept less than OP cut off by $\mathscr{C}_2$ or $\mathscr{C}_3$, then $\mathscr{L}'$ cannot exist. This condition is seen to be satisfied if P lies in $[A_1, F_1, I_1]$ or $[B_1, F_1, H_1]$. From this fact theorem 2 follows for R of type III also (see figure 5).

For R of type IV there is still the possibility that P lies in $[H_1, F_1, I_1]$. A sufficient condition to exclude this is seen to be provided by theorem 3. However, in this case there may be an $\mathscr{L}'$ with points on $\mathscr{C}$, the boundary of R. But this does not contradict theorem 3.

In Lemma 17 a result is proved which may be useful in deciding, in the case of bounded R, whether $\mathscr{L}_1$ and $\mathscr{L}_2$ are admissible or not.

10. Proofs of theorems 2 and 3

In the proofs the following notation will be adopted.

If points P and Q lie on $\mathscr{C}$, then $[P, Q]$ will denote the closed region bounded by the straight line PQ and the arc PQ of $\mathscr{C}$. Also $[P, S, Q]$ will denote the closed region bounded by straight lines PS and SQ and the arc PQ of $\mathscr{C}$, and (P, Q, R, S) the closed region lying between the straight lines PQ, QR, RS and the part of $\mathscr{C}$ intercepted in the quadrilateral $PQRS$. For example, the shaded region in the figure is to be denoted by (P, Q, R, S).

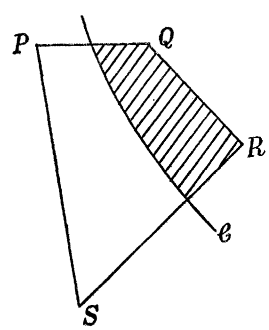

Figure 7

It will throughout be assumed that any A_i or B_i lying in these regions is excluded from them.

Suppose that there exists an R-admissible lattice $\mathscr{L}'$, *different from* $\mathscr{L}_1$ *and* $\mathscr{L}_2$, *but with its determinant equal to* Δ.

LEMMA 6. *Any* A_i *or* B_i *cannot belong to* $\mathscr{L}'$.

Proof. Suppose, for example, that A_1 belongs to $\mathscr{L}'$. Then A_1 must be primitive, for, otherwise, $\mathscr{L}'$ has a point inside R. The lines A_2A_3 (in case II, figure 4) and A_2C_1 (in cases III and IV, figures 5 and 6) are both equal and parallel to OA_1, and at a distance Δ/OA_1, from it. Therefore, by lemma 5, $\mathscr{L}'$ has a point inside R, unless A_2 belongs to $\mathscr{L}'$, in which case $\mathscr{L}'$ is identical with $\mathscr{L}_1$. This gives a contradiction.

LEMMA 7. *In case II (figure 4), none of the regions* $[A_1, H_1]$, $[A_2, H_2]$, …. $[A_6, H_6]$, $[B_1, I_1]$ $[B_2, I_2]$, …, $[B_6, I_6]$ *contains a point of* $\mathscr{L}'$.

Proof. Call these regions 1, 3, …, 11, 2, 4, …, 12 respectively as in figure 4.

Suppose 1 has a point P of $\mathscr{L}'$. The area of the triangle OB_1D_1 equals $\frac{1}{2}\Delta$. Both $[A_1, B_1]$ and 1 lie inside this triangle. Therefore, by lemmas 4 and 6, $[A_1, B_1]$ contains no point of $\mathscr{L}'$. As the area of the parallelogram $OA_6C_6A_1$ equals Δ and P does not coincide with A_1, it follows from lemma 3 that $[C_6, A_6]$ contains no point of $\mathscr{L}'$.

Now join A_1B_1 and A_4B_4 and let them intersect A_5A_6 in the points K_1, L_4 and A_2A_3 in L_1 and K_4 respectively (see figure 4; K_1L_1 and K_4L_4 are shown as broken lines.) Now $K_1L_1K_4L_4$ is a parallelogram, symmetrical about O, and of area 4Δ, since the area of $A_1K_1L_4A_4$ equals the area of $A_1C_6A_5A_4$ and the area of $A_1L_1K_4A_4$ equals the area of $A_1A_2C_3A_4$ (each by the theorem of parallelograms between the same parallels). Therefore, by lemma 2, $\mathscr{L}'$ has at least two points, other than O, in this parallelogram.

Now, by definition $\mathscr{L}'$ has no point in R. It has also been proved that it has no points in $[A_6, C_6]$ and $[A_1, B_1]$ and so also in their images in O. Therefore $\mathscr{L}'$ has a point in 3. Similarly, it can be proved that $\mathscr{L}'$ has a point in 5 and then in each of 7, 9 and 11 in turn. Therefore, application of lemma 3 to different parallelograms as in the beginning of the proof of this lemma shows that $\mathscr{L}'$ has no point in any of $[A_1, C_1]$, $[A_2, C_2]$, …, $[A_6, C_6]$.

Now consider the hexagon formed by the lines through the points A_i, perpendicular to the lines OA_i $(i = 1, …, 6)$. It was shown in §5 that the parts of this hexagon external to R lie entirely within the regions $[A_1, C_1]$, …, $[A_6, C_6]$ (see figure 4). As $\mathscr{L}'$ has been shown to have no point, besides O, inside R or in any of $[A_1, C_1]$, …, $[A_6, C_6]$, it follows that $\mathscr{L}'$ has no point in this hexagon.

But this is in contradiction to lemma 2, since this hexagon is convex, is symmetrical about O and has area $6OA_1^2 \tan 30° = 8(a^2+b^2+ab)/\sqrt{3} = 4\Delta$. Therefore the assumption is wrong, i.e. $\mathscr{L}'$ has no point in 1. Similarly, it can be proved that $\mathscr{L}'$ has no point in any of the regions 2, 3, …, 12, so that the lemma is true.

In the statements 'the point P lies below, above, to the right or to the left of a line Λ', the case when P lies on Λ will also be included.

In all the three cases II to IV define the points K_1, L_1, M_1 and N_1 to be the intersections, as shown in figure 3, of A_1B_1 with the lines A_6C_6, C_1A_2, B_6D_1 and D_2B_2, respectively. The points K_i, L_i, M_i, N_i $(i = 2, 3, 4, 5, 6)$ are defined by suitable rotations.

LEMMA 8. *If* $\mathscr{L}'$ *has a point in each of the regions* $[A_2, E_2, D_2]$, (B_2, E_2, K_3, M_3) and (A_3, E_3, N_2, L_2), *then* $\mathscr{L}'$ *must have a point either in* $[A_3, E_3, B_3]$ *or in* $[B_1, I_1]$; *the latter case arises only in cases III and IV.*

In the proof, without explicit mention, use will be made of the fact that for R of type II, $\mathscr{L}'$ has no point in any of the regions $[A_i, H_i]$ or $[B_i, I_i]$. It will then suffice in the proof to refer to figure 6 only, the relevant parts of which are reproduced here.

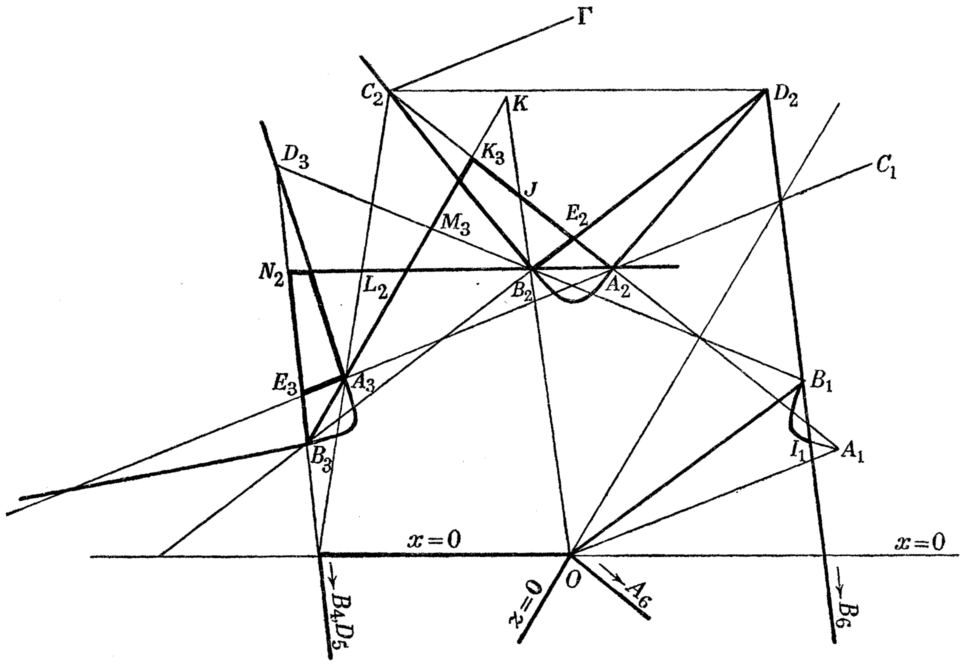

FIGURE 8

Proof. Call the points of $\mathscr{L}'$ in the regions $[A_2, E_2, D_2]$, (B_2, E_2, K_3, M_3) and (A_3, E_3, N_2, L_2) P_1, P_2 and P_3 respectively. Produce OB_2 to meet A_2C_2 in the point J, and M_3K_3 in K. (K may be below J, but that does not affect the argument.)

Then P_2 lies either (i) in the triangle B_2KM_3 or (ii) in the triangle B_2E_2J.

The lemma is proved for these cases separately.

(i) P_2 lies in the triangle B_2KM_3. Consider the point $P_3 - P_2$. Both P_2 and P_3 lie between the parallel lines OB_2 and $D_3B_3B_4D_5$. Therefore $P_3 - P_2$ lies to the right of the straight line $D_3B_3B_4D_5$, since O, P_2, P_3 and $P_3 - P_2$ are vertices of a parallelogram.

P_3 lies below A_2B_2 and P_2 above it. Therefore $P_3 - P_2$ lies below the x-axis, since A_2B_2 is parallel to the x-axis.

P_3 lies to the left of A_3B_3 and P_2 to its right. Therefore $P_3 - P_2$ lies to the left of the line $z = 0$, which is the line through O parallel to A_3B_3.

Consequently $P_3 - P_2$ lies in the triangle formed by the lines B_3B_4, $x = 0$ and $z = 0$ (see figure 9).

Therefore for R of type III or IV (figures 5 and 6), $P_3 - P_2$ lies in $[B_4, I_4]$. By symmetry, $P_2 - P_3$ lies in $[B_1, I_1]$, i.e. the lemma is true for this case.

For R of type II (figure 4), $P_3 - P_2$ lies in R since it has been proved in lemma 7 that $[B_4, I_4]$ contains no point of $\mathscr{L}'$. This contradicts the definition of $\mathscr{L}'$. Therefore (i) does not arise for these R.

(ii) The point P_2 lies in the triangle B_2E_2J. Consider $P_1 - P_2$. Both P_1 and P_2 lie between the parallel lines OB_2 and $B_6B_1D_2$. Therefore $P_1 - P_2$ lies to the left of B_1B_6, since OB_2 passes through the origin.

P_1 lies above the line $A_1A_2C_2$ and P_2 below it. Therefore P_1-P_2 lies above OA_6, which is the line through O parallel to A_1A_2.

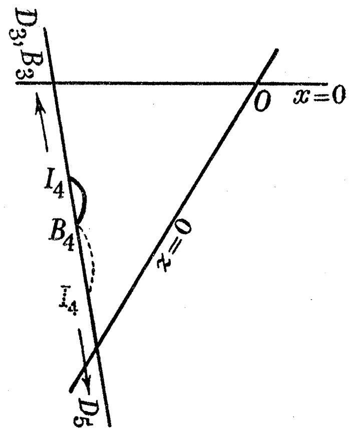

FIGURE 9

P_1 lies below D_2B_2 and P_2 above it. Therefore P_1-P_2 lies below OB_1, the line through O parallel to D_2B_2.

Let $C_2\Gamma$ be the line through C_2 parallel to A_2C_1. This line is above C_2D_2. Therefore both P_1 and P_2 lie between the parallel lines C_1A_2 and $C_2\Gamma$. Also the lines A_5A_6, OA_1, A_2C_1 and $C_2\Gamma$ are parallel and the distances between consecutive lines are the same. Therefore P_1-P_2 lies above A_5A_6.

From the above P_1-P_2 lies in the quadrilateral formed by OA_6, A_6E_6, E_6B_1 and B_1O (see figure 10).

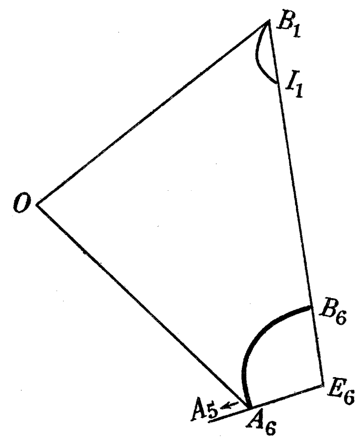

FIGURE 10

Therefore for R of type II P_1-P_2 lies in $[A_6, E_6, B_6]$. Hence P_2-P_1 lies in $[A_3, E_3, B_3]$, as asserted in this lemma. For R of type III or IV either P_1-P_2 lies in $[B_1, I_1]$ or P_2-P_1 lies in $[A_3, E_3, B_3]$.

This completes the proof of the lemma.

LEMMA 9. *$\mathscr{L}'$ has a point in one at least of the regions* $[A_1, E_1, B_1]$, $[A_2, E_2, B_2]$ *and* $[A_3, E_3, B_3]$.

Proof. (Either of figures 4 and 6 may be consulted.) Suppose $\mathscr{L}'$ does not have a point in any of the three regions. By symmetry it does not have a point in their images in O either.

Consider the parallelogram $M_1N_1M_4N_4$, shown by dotted lines in figure 6. Its area is 4Δ and it is symmetric about O. Therefore it contains a point of $\mathscr{L}'$ other than O.

Now, $\mathscr{L}'$ has no point in

(i) R, by definition of $\mathscr{L}'$,

(ii) $[A_1, B_1]$, since it lies in $[A_1, E_1, B_1]$,

(iii) $[A_2, E_2, B_2]$, by hypothesis,

(iv) $[B_3, I_3]$, by lemma 7 in case II, and since it lies in $[A_3, E_3, B_3]$ in cases III and IV, and

(v) the images of $[A_1, B_1]$, $[A_2, E_2, B_2]$ and $[B_3, I_3]$ in O.

Therefore, $\mathscr{L}'$ has a point in one at least of (A_2, E_2, N_1, L_1) and (A_5, E_5, N_4, L_4), and since they are images in O, in both.

Similarly, by considering parallelograms $K_3L_6K_6L_6$ and $M_2N_2M_5N_5$ it can be shown that $\mathscr{L}'$ has a point in each of (B_2, E_2, K_3, M_3) and (A_3, E_3, N_2, L_2).

Therefore it follows from lemma 8 that $\mathscr{L}'$ has a point in $[A_3, E_3, B_3]$ or, in the case of R of type III or IV only, in $[B_1, I_1]$, which then lies in $[A_1, E_1, B_1]$. That gives a contradiction and hence the lemma.

Without loss of generality, *it can now be supposed that* $\mathscr{L}'$ *has a point* P' *in* $[A_1, E_1, B_1]$. Then

LEMMA 10. $\mathscr{L}'$ *has no point in* $[A_2, E_2, B_2]$ *or* $[A_3, E_3, B_3]$.

Proof. (Consult figures 4 or 6.) Suppose $\mathscr{L}'$ has a point P_2 in $[A_2, E_2, B_2]$. Consider $P'+P_2$. P' lies to the left of the line A_1C_1 and P_2 to the left of OA_2, which is parallel to A_1C_1. Therefore $P'+P_2$ lies to the left of A_1C_1.

P_2 lies below B_2D_2 and P' below the parallel line OB_1. Therefore $P'+P_2$ lies below B_2D_2.

As both P' and P_2 lie above OA_1 and to the right of OB_2, $P'+P_2$ also lies above OA_1 and to the right of OB_2.

Consequently, since $P'+P_2$ does not lie in R, it lies either in $[A_1, C_1]$ or in $[B_2, D_2]$.

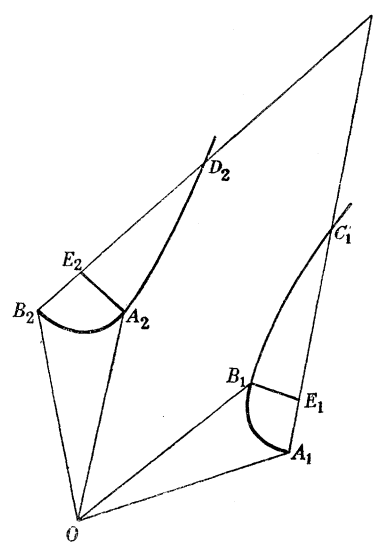

FIGURE 11

Now the area of the triangle OA_1C_1 is equal to $\frac{1}{2}\Delta$. P' lies in OA_1C_1 and is different from A_1 and C_1. Also the point $P'+P_2$ is not collinear with O and P'. Consequently by lemma 4, $P'+P_2$ does not lie in $[A_1, C_1]$. Similarly, $P'+P_2$ does not lie in $[B_2, D_2]$. Hence the contradiction. Therefore $\mathscr{L}'$ has no point in $[A_2, E_2, B_2]$.

If $\mathscr{L}'$ is supposed to have a point P_3 in $[A_3, E_3, B_3]$, a contradiction can be obtained by considering the point $P_3 - P'$ in a similar way.

Therefore the lemma is true, i.e. $\mathscr{L}'$ has no point in $[A_2, E_2, B_2]$ or $[A_3, E_3, B_3]$.

10·1. *End of the proof for R of type II*

Let R be of type II. (Consult figure 4.)

By lemma 10 the lattice $\mathscr{L}'$ has no point in $[A_2, E_2, B_2]$ or $[A_3, E_3, B_3]$, and hence, by taking images in O, none also in $[A_5, E_5, B_5]$ or $[A_6, E_6, B_6]$. Also by lemma 7, it has no point in 2 or 8.

Now consider the parallelogram $D_2B_3D_5B_6$, shown with dotted lines in the figure. It satisfies all the conditions of lemma 2 and hence contains two points $\pm P$ of $\mathscr{L}'$. As $\mathscr{L}'$ has no point in R, $[A_2, E_2, B_2]$, $[A_5, E_5, B_5]$, 2 or 8, one of the points $\pm P$, say P, lies in $[A_2, E_2, D_2]$.

Similarly, by considering the parallelograms $M_2N_2M_5N_5$ and $K_3L_3K_6L_6$, it is seen that a point of $\mathscr{L}'$ lies in each of (A_3, E_3, N_2, L_2) and (B_2, E_2, K_3, M_3).

Hence lemma 8 applies, so that $\mathscr{L}'$ has a point in $[A_3, E_3, B_3]$. But this contradicts lemma 10. Therefore the assumption at the beginning of §10 was wrong, i.e. there is no lattice, other than $\mathscr{L}_1$ and $\mathscr{L}_2$, of determinant Δ, which does not have a point inside R.

This completes the proof of theorem 2 when R is of type II.

10·2. *Proofs of theorems* 2 *and* 3 (*continued*)

Hereafter R will belong only to types III or IV.

LEMMA 11. *$\mathscr{L}'$ has a point in* $[B_1, F_1, A_1]$.

Proof. (See figure 6.) Suppose $\mathscr{L}'$ has no point in $[B_1, F_1, A_1]$. By symmetry $\mathscr{L}'$ has no point in $[B_4, F_4, A_4]$. Applying lemma 2 to the paralleogram $D_2B_3D_5B_6$, one finds that $\mathscr{L}'$ has two points $\pm P$, other than O, in this parallelogram.

Now $\mathscr{L}'$ has no point in

(i) $[B_1, I_1]$, since it lies in $[B_1, F_1, A_1]$,

(ii) $[A_2, E_2, B_2]$, by lemma 10,

(iii) $[B_3, I_3]$, since it lies in $[B_3, E_3, A_3]$; and so also in

(iv) $[B_4, I_4]$, $[A_5, E_5, B_5]$ and $[B_6, I_6]$.

Therefore, one of these points $\pm P$ lies in $[A_2, E_2, D_2]$.

Similarly, by considering the parallelograms $M_2N_2M_5N_5$ and $K_3L_3K_6L_6$ it can be shown that $\mathscr{L}'$ has points in each of (A_3, E_3, N_2, L_2) and (B_2, E_2, K_3, M_3).

Therefore by lemma 8, $\mathscr{L}'$ has a point in $[B_1, I_1]$ which is contained in $[B_1, F_1, A_1]$. But this is a contradiction so that the lemma is true, i.e. $\mathscr{L}'$ has a point in $[B_1, F_1, A_1]$.

10·3. *Two lemmas*

Let the point of $\mathscr{L}'$ in $[B_1, F_1, A_1]$ be denoted by P. By taking, if necessary, the lattice point on OP nearest to P, P can be supposed to be primitive.

LEMMA 12. *Any line parallel to OP has a length greater than* $2OP$ *intercepted between* (*i*) A_1C_1 *and* C_2A_3, (*ii*) B_2D_3 *and* B_4D_4 (*see figure* 6).

Proof. (i) P lies between OA_2 and A_1C_1. Also A_1C_1 and C_2A_3 are lines parallel to OA_2 and at equal distances on opposite sides.

(ii) Similarly.

LEMMA 13. (*a*) (*i*) *If the line* Λ, *parallel to* OP *and at a distance* Δ/OP *above it, has a length less than* OP *intercepted by* $\mathscr{C}_2$ *of* $\mathscr{C}_3$, *there is a contradiction to the hypothesis that* $\mathscr{L}'$ *has no point inside* R.

(ii) *If the line* Λ', *parallel to* OP *and at a distance* Δ/OP *below it, has a length less than* OP *intercepted by* $\mathscr{C}_4$ *or* $\mathscr{C}_5$, *then there is a contradiction to the hypothesis about* $\mathscr{L}'$.

(*b*) *If in the above one of the two intercepts equals* OP *and the other is greater than or equal to* OP, *then* $\mathscr{L}'$ *may be admissible.*

Proof. (*a*) (i) Suppose, for example, that Λ has a length less than OP intercepted by $\mathscr{C}_2$.

By lemma 12, the length of Λ, intercepted between the lines A_1C_1 and C_2A_3 is greater than $2OP$. Also Λ is at a distance Δ/OP from OP. Therefore, by lemma 5, at least two points of $\mathscr{L}'$ lie on the segment Λ_1 of Λ intercepted between A_1C_1 and C_2A_3.

The length of the segment of Λ, intercepted by $\mathscr{C}_2$, is less than OP. Therefore at most one point of $\mathscr{L}'$ lies on this part of Λ.

No point of $\mathscr{L}'$ lies in the part of Λ_1 intercepted between $\mathscr{C}_1$ and A_1C_1, because $[A_1, C_1]$ does not contain any point of $\mathscr{L}'$, other than P. Nor is there any point of $\mathscr{L}'$ in the part, if any, of Λ_1 between $\mathscr{C}_3$ and C_2A_3, for $\mathscr{L}'$ has no point in $[A_3, E_3, B_3]$.

Consequently at least one point of $\mathscr{L}'$ lies in the part of Λ_1 lying inside R. This gives the required contradiction.

(ii) Similarly.

(*b*) If $\mathscr{L}'$ has no point in the part of Λ_1 lying inside R, two points of $\mathscr{L}'$ must lie on the part of Λ_1 intercepted by $\mathscr{C}_2$ or $\mathscr{C}_3$. This is possible if the conditions of the lemma are satisfied and the end-points of the intercept equal to OP are points of $\mathscr{L}'$. In this case $\mathscr{L}'$ may be admissible.

10·4. *End of the proof of theorem* 2

Theorem 2 for R of type II has already been proved. Therefore in this section only R of type III will be considered.

In lemma 11 it was seen that for such R $\mathscr{L}'$ has a point P in $[A_1, F_1, B_1]$. In this case, $[A_1, F_1, B_1]$ consists of two parts, namely, $[A_1, H_1]$ and $[B_1, I_1]$ (see figure 5).

In lemma 14 below it will be shown that P cannot lie in $[A_1, H_1]$ or in $[B_1, I_1]$. This gives a contradiction, so that for R of type III, too, all lattices $\mathscr{L}'$, of determinant Δ and different from $\mathscr{L}_1$ and $\mathscr{L}_2$, have a point other than O inside R, i.e. theorem 2 is true.

LEMMA 14. *P cannot lie* (*i*) *in* $[A_1, H_1]$, (*ii*) *in* $[B_1, I_1]$.

Proof (see figure 5). (i) P lies in $[A_1, H_1]$. Join OP and let it intersect D_2B_1 at Q. From D_2B_1 cut off D_2Q' equal to B_1Q. Join B_2Q'.

As OB_1 is equal and parallel to B_2D_2, the triangle OB_1Q is congruent to B_2D_2Q'. Therefore B_2Q' is equal and parallel to OQ. Hence, the area of the parallelogram $OB_2Q'Q$ = area of $OB_2D_2B_1 = \Delta$. Therefore the distance of B_2Q' from OP equals Δ/OQ, which is greater than or equal to Δ/OP.

Now B_2Q' lies in the triangle $B_2M_2D_2$, for OQ lies in the triangle formed by OB_1, B_1I_1 and the x-axis. Also as the line B_1D_2, being parallel to OB_2, intersects $\mathscr{C}_2$ only at D_2, Q' lies inside R. Consequently B_2Q' meets $\mathscr{C}_2$ in a point lying between B_2 and Q'. Therefore, the length of B_2Q' intercepted by $\mathscr{C}_2$ is less than $B_2Q' = OQ \leqslant OP$.

As the line Λ is nearer to O than B_2Q', the length of its intercept on $\mathscr{C}_2$ is less than OP, since, as a line moves towards O parallel to itself, its length intercepted by $\mathscr{C}_2$ does not increase.

Hence, by lemma 13, $\mathscr{L}'$ has a point inside R. This is a contradiction. Therefore P does not lie in $[A_1, H_1]$.

(ii) The proof follows as in (i).

10·5. *End of the proof of theorem* 3

Henceforth R belongs to type IV and reference is made only to figure 6.

LEMMA 15. *P cannot lie (i) in $[A_1, F_1, I_1]$, (ii) in $[B_1, F_1, H_1]$.*

Proof. (i) Let P lie in $[A_1, F_1, I_1]$. Join OP and let it intersect $B_1 I_1$ in Q. Then the proof follows by repeating the argument of lemma 15, word by word.

(ii) Similarly.

End of proof. P lies in $[A_1, F_1, B_1]$ but not in $[A_1, F_1, I_1]$ or in $[B_1, F_1, H_1]$. Therefore P lies in $[H_1, F_1, I_1]$.

Let OP intersect $\mathscr{C}_1$ at Q_1. Then $OQ_1 \leqslant OP$ and $\Delta/OQ_1 \geqslant \Delta/OP$. Therefore, if it can be proved that the line parallel to OQ_1, at a distance Δ/OQ_1 from it, has a length less than or equal to OQ_1 intercepted by $\mathscr{C}_2$, it will follow that Λ, the line parallel to OP at a distance Δ/OP, has a length less than or equal to OP intercepted by $\mathscr{C}_2$. Then lemma 13 will lead to a contradiction to the definition of $\mathscr{L}'$.

Similar results hold for $\mathscr{C}_3$, $\mathscr{C}_5$ and $\mathscr{C}_6$.

This can be summed up as follows:

Let P be any point on $\mathscr{C}_1$ between H_1 and I_1. If the line parallel to OP and at a distance Δ/OP above (or below) OP has a length less than OP intercepted by one at least of $\mathscr{C}_2$ and $\mathscr{C}_3$ (or $\mathscr{C}_5$ and $\mathscr{C}_6$), then no lattice, different from $\mathscr{L}_1$ and $\mathscr{L}_2$, and of determinant Δ, is admissible for R.

$\mathscr{L}_1$ and $\mathscr{L}_2$ have points on the boundary of R and may be admissible.

If for some P, the least of the above intercepts is OP, then the lattice $\mathscr{L}'$, generated by P and an end-point of this intercept, is admissible if $\mathscr{L}'$ has no point other than O, inside R.

In lemma 16 it will be proved that there exist lines $P_1 P_2$ and $P_3 P_4$ equal and parallel to OP such that P_1 and P_2 lie on $\mathscr{C}_2$ while P_3 and P_4 lie on $\mathscr{C}_3$. Then the equivalence of the above with theorem 3 follows from the fact that, if a line moves towards O parallel to itself, its intercept on $\mathscr{C}_2$ or $\mathscr{C}_3$ does not increase.

10·6. *The existence of $P_1 P_2$ and $P_3 P_4$*

LEMMA 16. *For all points P on $\mathscr{C}_1$ between H_1 and I_1 there exists*

(i) a line $P_1 P_2$, equal and parallel to OP with P_1 and P_2 on $\mathscr{C}_2$, and

(ii) a line $P_3 P_4$ equal and parallel to OP with P_3 and P_4 on $\mathscr{C}_3$.

Proof. (i) As a line Λ moves towards O parallel to itself, its intercept by $\mathscr{C}_2$, if finite at any stage, decreases continuously to zero, except when $\mathscr{C}_2$ contains a line Λ' parallel to Λ. Then the intercept decreases continuously to the length of Λ' when Λ coincides with Λ'; it becomes zero after that.

If $\mathscr{C}_2$ contains a line Λ', parallel to OP and of length not less than OP, $P_1 P_2$ can be taken to be any segment of length OP on Λ', i.e. the lemma holds. Therefore, there remains here only the case where $\Lambda' < OP$.

It will suffice to prove the lemma for all points P on the line $B_1 I_1$. For, if P does not lie on $B_1 I_1$, OP can be produced to meet $B_1 I_1$ at Q. Then $Q_1 Q_2$ can be found equal and parallel to OQ with Q_1 and Q_2 on $\mathscr{C}_2$. Then moving $Q_1 Q_2$ parallel to itself towards O continuously one arrives at a position where its intercept P_1P_2 on $\mathscr{C}_2$ equals OP. In the rest of the proof of the lemma P will therefore be supposed to lie on B_1I_1.

Produce OB_2 to B. From B_2B cut off $B_2P'' = B_1P$. P'' lies outside R and below C_2D_2, since, for the x-co-ordinates $x(P)$, etc., it is clear that

$$x(P'')-x(B_2) = x(B_1)-x(P) \leqslant x(B_1) = x(D_2)-x(B_2).$$

Join D_2P''. As D_2B_2 is equal and parallel to OB_1, the triangles OB_1P and D_2B_2P'' are congruent. Therefore D_2P'' is equal and parallel to OP. Also D_2P'' lies entirely inside $[C_2, D_2]$.

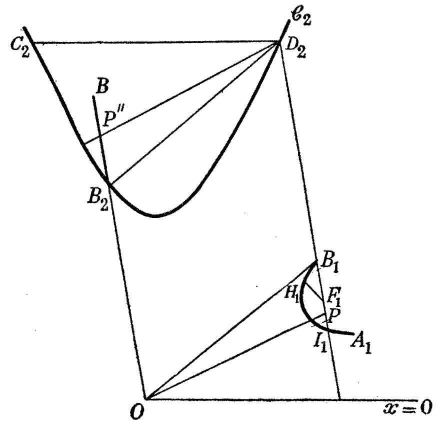

FIGURE 12

Therefore producing D_2P'' to intersect $\mathscr{C}_2$ one gets a line parallel to OP with a length greater than or equal to OP intercepted by $\mathscr{C}_2$. This line can be moved continuously parallel to itself towards O, till its intercept P_1P_2 on $\mathscr{C}_2$ equals OP. This completes the proof of (i).

(ii) Similarly.

With the proof of this lemma, the proofs of theorems 2 and 3 are complete.

11. Some remarks on the admissibility of $\mathscr{L}_1$ and $\mathscr{L}_2$

In this section R may belong to any of the types II to IV.

By the symmetry of R it is obvious that either both or none of $\mathscr{L}_1$ and $\mathscr{L}_2$ is admissible. Therefore, attention will be confined to $\mathscr{L}_1$ only.

In vector notation any point of $\mathscr{L}_1$ is of the form $\xi A_1 + \eta A_2$, where ξ and η are integers. Therefore, from table 1, it is clear that the co-ordinates of a point of $\mathscr{L}_1$ are given by

$$x = a\xi + (a+b)\eta, \quad y = -(a+b)\xi - b\eta, \quad z = b\xi - a\eta. \tag{11·1}$$

Suppose a/b is rational. Then $\mathscr{L}_1$ has an infinity of points on the line $x = 0$. Therefore, if R is unbounded, $\mathscr{L}_1$ cannot be R-admissible. In this case the problem of finding the critical determinant of R remains unsolved, unless, of course, R belongs to type IV and theorem 3 gives a critical lattice different from $\mathscr{L}_1$ and $\mathscr{L}_2$. If R is bounded $\mathscr{L}_1$ may still be admissible. Both now and also when a/b is irrational lemma 17 gives a criterion for the admissibility of $\mathscr{L}_1$ for bounded R.

LEMMA 17. *Let D be an integer such that*

$$f[0, -\surd\{(a^2+ab+b^2)(D+1)\}, \surd\{(a^2+ab+b^2)(D+1)\}] \geqslant f(c, -2c, c). \tag{11·2}$$

Then $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *are admissible if, and only if, the co-ordinates of the finite number of lattice points* (11·1) *with*

$$0 < \xi^2 + \xi\eta + \eta^2 \leqslant D, \tag{11·3}$$

satisfy the relation

$$f(x, y, z) \geqslant f(c, -2c, c). \tag{11·4}$$

Proof. V_1, the point with co-ordinates $[0, -\surd\{(a^2+ab+b^2)(D+1)\}, \surd\{(a^2+ab+b^2)(D+1)\}]$, is at a distance $\{\frac{4}{3}(a^2+ab+b^2)(D+1)\}^{\frac{1}{2}}$ from O. By the symmetry and convexity conditions R lies within the hexagon $V_1V_2V_3V_4V_5V_6$ and so also in its circumcircle:

$$x^2 + \tfrac{1}{3}(z-y)^2 = \tfrac{4}{3}(a^2+b^2+ab)(D+1).$$

The points of $\mathscr{L}_1$ within this circle satisfy the relation

$$\xi^2 + \xi\eta + \eta^2 \leqslant D,$$

and the lemma follows from (11·3) and (11·4).

If R is unbounded and a/b is irrational, it may not be easy to decide whether $\mathscr{L}_1$ and $\mathscr{L}_2$ are admissible or not. Here, to prove that all the points of $\mathscr{L}_1$ lie outside or on the boundary of R, it must be shown that if $f(x, y, z) = f\{a\xi + (a+b)\eta, -(a+b)\xi - b\eta, -b\xi - a\} = \phi(\xi, \eta)$, then $\phi(\xi, \eta) \geqslant f(c, -2c, c)$ for all integers ξ, η, not both zero. This may be an extremely difficult problem. However, if $\phi(\xi, \eta)$ is a polynomial with rational coefficients, elementary considerations, like those of congruences or irrationality of roots of an equation, may give an easy solution of the problem.

PART II. APPLICATIONS

12. Preliminary remarks

In order to apply the results of part I to any particular region R one has first to verify that R satisfies conditions (2·1) to (2·8). One has then to examine whether the relation (4·1) is satisfied or not. If it is not, then theorem 1 applies, and gives both the critical determinant and the only critical lattice of R. If (4·1) is satisfied, one determines the number a, b and Δ. In practice it may suffice only to write down the equations giving these quantities. One has next to investigate if R belongs to types II and III dealt with in theorem 2 or to type IV to which theorem 3 applies.

As has already been seen in § 8, R belongs to types II or III if the point

$$F_2\left(\frac{2(a^2+ab+b^2)}{(2b+a)}, -\frac{(a^2+ab+b^2)}{(2b+a)}, -\frac{(a^2+ab+b^2)}{(2b+a)}\right)$$

lies below (or on) the point $T_2(2c, -c, -c)$, i.e. if

$$a^2 + b^2 + ab \leqslant c(2b+a). \tag{12·1}$$

Similarly, R belongs to type IV if

$$a^2 + b^2 + ab > c(2b+a). \tag{12·2}$$

In determining the type to which R belongs it is often convenient to use the following:

R belongs to type II if, at the point A_1, the function $f(x, y, z)$ decreases as (x, y, z) moves along A_1A_2 in the direction of increasing x. For this implies that the points in the neighbourhood of A_1 and lying between A_1 and A_2 are inner points of R.

Now if R belongs to the first set, i.e. to types II or III, theorem 2 applies and gives $\Delta(R)$ if $\mathscr{L}_1$ and $\mathscr{L}_2$ are admissible.

If R belongs to type IV, theorem 3 applies. To obtain any result from this theorem one has first to show that for all points P on the boundary $\mathscr{C}$ between I_1 and $T_1(c, -2c, c)$ the line Λ, parallel to and at a distance Δ/OP from OP has a length less than or equal to OP intercepted by $\mathscr{C}_2$ or $\mathscr{C}_3$. In practice it is convenient to take P between A_1 and T_1. It may often be more convenient to prove the equivalent area condition, viz. the area of one at least of the parallelograms OPP_1P_2 and OPP_3P_4 is not less than Δ. P_1P_2 and P_3P_4 were defined in lemma 16 as also in theorem 3. This condition can be expressed analytically as follows:

If (x_1, y_1, z_1) is a point between A_1 and T_1, i.e. if

$$f(x_1, y_1, z_1) = f(c, -2c, c), \tag{12·3}$$

where
$$a \leqslant x_1 \leqslant c, \quad c \leqslant z_1 \leqslant b, \tag{12·4}$$

and if
$$f(p \pm \tfrac{1}{2}x_1, q \pm \tfrac{1}{2}y_1, r \pm \tfrac{1}{2}z_1) = f(c, -2c, c), \tag{12·5}$$

and both the points $(p \pm \tfrac{1}{2}x_1, q \pm \tfrac{1}{2}y_1, r \pm \tfrac{1}{2}z_1)$ lie either in the second or in the third sector then,

$$|p(z_1 - y_1) - x_1(r - q)| \geqslant \Delta\sqrt{3}. \tag{12·6}$$

If the equality sign in (12·6) is not needed for any point P different from A_1, then theorem 3 gives the value of $\Delta(R)$ provided that $\mathscr{L}_1$ and $\mathscr{L}_2$ are admissible. In this case $\mathscr{L}_1$ and $\mathscr{L}_2$ are the only critical lattices. If the equality sign in (12·6) is necessary for some P different from A_1 and if the corresponding lattice $\mathscr{L}$ is admissible, then $\Delta(R)$ is equal to Δ even if $\mathscr{L}_1$ and $\mathscr{L}_2$ are not admissible. Obvious transformations applied to $\mathscr{L}$ give six critical lattices, except when P coincides with T_1, when there may be only three critical lattices related to $\mathscr{L}$.

If in (12·6) the sign of equality is needed for P coinciding with T_1, then, because of the symmetry of R, the areas of both OPP_1P_2 and OPP_3P_4 will be equal to Δ and the points P_1, P_2, P_3 and P_4 will lie on the same straight line, say in the order given here. In this case it can sometimes be proved that the lattice $\mathscr{L}''$ generated by P and P_1 is not critical by the following

Lemma 18. *$\mathscr{L}''$ cannot be critical unless $P_1P_2 = P_2P_3 = P_3P_4 = OP$.*

Proof. From lemma 13 (b) it is easy to see that if $\mathscr{L}''$ is critical, then P_1, P_2, P_3 and P_4 all belong to $\mathscr{L}''$ and the line segment P_1P_4 contains no other point of $\mathscr{L}''$.

One way of proving that the above conditions are not satisfied is to show that the point $P_2 - P$ does not lie on $\mathscr{C}$, the boundary of R. This gives the

Corollary. *If $P_2 - P$ does not lie on $\mathscr{C}$, $\mathscr{L}''$ cannot be critical.*

Sometimes it may be more convenient to work in the second sector instead of the first.

13. The region $|r^3 \sin 3\theta| \leqslant 8c^3$

In the co-ordinate system defined in §2 the region R can be defined by the relation

$$f(x, y, z) = |xyz| \leqslant 2c^3.$$

Theorem 4. *Let R be the region defined by*

$$f(x, y, z) = |xyz| \leqslant 2c^3, \quad x + y + z = 0. \tag{13·1}$$

Define a, b and Δ by the relations

$$2c^3 = ab(a+b) = (b-a)(a+2b)(2a+b) \tag{13·2}$$

and

$$\Delta = \frac{2}{\sqrt{3}}(a^2+b^2+ab). \tag{13·3}$$

Then $\Delta(R) = \Delta$, *and the only critical lattices are* $\mathscr{L}_1$ *and* $\mathscr{L}_2$, *where*

(i) $\mathscr{L}_1$ *is generated by the points* $(a, -a-b, b)$ *and* $(a+b, -b, -a)$, *and*

(ii) $\mathscr{L}_2$ *is generated by* $(b, -a-b, a)$ *and* $(a+b, -a, -b)$.

Proof. It can easily be verified that R satisfies the conditions (2·1) to (2·8) and (4·1).

The equations (4·2), which define a and b in part I, are equivalent to (13·2) and give

$$b^3+b^2a-2a^2b-a^3 = 0. \tag{13·4}$$

It will now be shown that R belongs to type II. Write for $x \geqslant 0$, $z \geqslant 0$,

$$f(x,y,z) = xz(x+z).$$

The equation of A_1A_2 is

$$a^2+b^2+ab = ax-by = (a+b)x+bz \quad (\text{see } \S 5).$$

Therefore, along this line,

$$\frac{df}{dx} = (2xz+z^2)+(x^2+2xz)\frac{dz}{dx}$$

$$= \frac{1}{b}\{b(2xz+z^2)-(a+b)(x^2+2xz)\}.$$

At the point $A_1(a, -a-b, b)$ the above gives

$$\frac{df}{dx} = \frac{1}{b}\{b(b^2+2ab)-(a+b)(a^2+2ab)\}$$

$$= \frac{1}{b}\{b^3-a^3-3a^2b\}$$

$$= -a(a+b) \quad (\text{by } (13·4))$$

$$< 0.$$

Therefore R belongs to type II.

Consequently theorem 2 applies and theorem 4 follows if it can be shown that $\mathscr{L}_1$ is admissible.

Now any point of $\mathscr{L}_1$ has co-ordinates

$$X = a\xi+(a+b)\eta, \quad Y = -(a+b)\xi-b\eta, \quad Z = b\xi-a\eta,$$

with integers ξ, η. Thus

$$\begin{aligned} f(X,Y,Z) &= |\{a\xi+(a+b)\eta\}\{(a+b)\xi+b\eta\}\{b\xi-a\eta\}| \\ &= |ab(a+b)(\xi^3-\eta^3)+(b^3-a^3+3ab^2)\xi^2\eta+(b^3-a^3-3a^2b)\xi\eta^2| \\ &= ab(a+b)|\xi^3-\eta^3+2\xi^2\eta-\xi\eta^2| \quad (\text{by } (13·4)). \end{aligned}$$

As the equation

$$\xi^3+2\xi^2\eta-\xi\eta^2-\eta^3 = 0$$

gives no rational value for ξ/η,

$$|\xi^2+2\xi^2\eta-\xi\eta^2-\eta^3| \geqslant 1$$

for all integers ξ, η, not both zero. Therefore for all points of $\mathscr{L}_1$, other than 0,

$$f(X, Y, Z) \geqslant ab(a+b),$$

i.e. $\mathscr{L}_1$ is admissible and the theorem follows.

Theorem 4 is easily seen to be equivalent to the following theorem of Mordell (1943 *a*, *b*):

THEOREM. *If* $f(x, y) = ax^3+bx^2y+cxy^2+dy^3$, *and if*

$$D = 18abcd-27a^2d^2+b^2c^2-4ac^3-4db^3>0,$$

then integer values of x, y, not both zero, exist such that

$$|f(x,y)| \leqslant \sqrt[4]{\frac{D}{49}}. \tag{13·5}$$

The equality in (13·5) *is necessary if and only if*

$$e^{-1}f(x,y) \sim x^3+x^2y-2xy^2-y^3,$$

where e is any constant.

Other proofs of this theorem have been given by Davenport (1945) and Delaunay (1945).

14. Circular hexagons

THEOREM 5. *Let R be a region with hexagonal symmetry bounded by circular arcs. Let the boundary arc in the first sector meet the lines* $x = 0$, $z = 0$ *at an angle* $\sec^{-1}\lambda$, *where* $1 \leqslant \lambda \leqslant 2$, *and let the circle, of which this arc is a part, have its centre at the point* $(g, -2g, g)$. *Then* $\Delta(R) = 2c^2\sqrt{3}$, *where* $c = \frac{1}{2}(2-\lambda)\,g$, *and there is only one critical lattice, namely, that generated by the middle points of the boundary arcs.*

Proof. The first sector boundary of R is easily seen to have the equation

$$(x-g)^2+\tfrac{1}{3}\{(z-g)-(y+2g)\}^2 = \lambda^2g^2, \quad x+y+z=0.$$

Eliminating y and solving for g,

$$3(4-\lambda^2)\,g = 6(x+z) \pm 2\{9(x+z)^2-(12-3\lambda^2)\,(x^2+xz+z^2)\}^{\frac{1}{2}}.$$

On considering the intersection of the above with the line $x=0$ one finds that the positive value of the radical is to be taken. Therefore, in the sector $x \geqslant 0$, $z \geqslant 0$ the equation of $\mathscr{C}$, the boundary of R is given by

$$f(x,y,z) = 6(x+z)+2\{9(x+z)^2-(12-3\lambda^2)\,(x^2+xz+z^2)\}^{\frac{1}{2}} = 3(4-\lambda^2)\,g. \tag{14·1}$$

The middle point of this arc is $(c, -2c, c)$, where

$$3(4-\lambda^2)\,g = 12c+6c\lambda,$$

or

$$c = \tfrac{1}{2}(2-\lambda)\,g.$$

Now $$f(0, -3c, 3c) = 6c\{3+(3\lambda^2-3)^{\frac{1}{2}}\} = 3(2-\lambda)\,\{3+(3\lambda^2-3)^{\frac{1}{2}}\}\,g,$$

and $$f(c, -2c, c) = 3(4-\lambda^2)\,g.$$

Therefore $$f(0, -3c, 3c) \geqslant f(c, -2c, c),$$

since $$3+(3\lambda^2-3)^{\frac{1}{2}} \geqslant 2+\lambda,$$

because $$3\lambda^2-3-(\lambda-1)^2 = 2(\lambda+2)\,(\lambda-1) \geqslant 0.$$

Consequently the region belongs to type I and the theorem follows on application of theorem 1.

15. Parabolic hexagons

Theorem 6. *Let R be a region with hexagonal symmetry bounded by parabolic arcs. Let the mid-point of the arc in the first sector be $(c, -2c, c)$, and suppose the latus rectum of the parabola, of which this arc forms a part, is λc, where $\lambda \geq \frac{8}{3}$. Then*

(*a*) *if $\lambda \geq 3$, $\Delta(R) = \Delta$, and the lattice generated by the mid-points of the boundary arcs is the only critical lattice;*

(*b*) *if $\frac{8}{3} \leq \lambda < 3$, define a, b and Δ by*

$$12\lambda c = 3\lambda(a+b) + \{9\lambda^2(a+b)^2 - 24\lambda(b-a)^2\}^{\frac{1}{2}}$$
$$= 3\lambda(a+2b) + \{9\lambda^2(a+2b)^2 - 216\lambda a^2)\}^{\frac{1}{2}}, \tag{15·1}$$

and
$$\Delta = \frac{2}{\sqrt{3}}(a^2+ab+b^2), \tag{15·2}$$

then $\Delta(R) = \Delta$ and there are just two critical lattices.

Proof. The equation of the boundary of R in the first sector is easily seen to be

$$(z-x)^2 = -3\lambda c(y+2c),$$

i.e.
$$6\lambda c^2 - 3\lambda c(x+z) + (z-x)^2 = 0.$$

Solving for c, one gets

$$12\lambda c = 3\lambda(x+z) \pm \{9\lambda^2(x+z)^2 - 24\lambda(z-x)^2\}^{\frac{1}{2}}.$$

On considering the intersection of this arc with the line $x = 0$, one finds that the positive value of the radical is to be taken, so that the equation of the boundary of R in the first sector can be expressed as

$$f(x,y,z) = 3\lambda(x+z) + \{9\lambda^2(x+z)^2 - 24\lambda(z-x)^2\}^{\frac{1}{2}} = 12\lambda c. \tag{15·3}$$

Now the two parts of the theorem will be proved one by one.

(*a*) Let $\lambda \geq 3$. Then
$$f(0, -3c, 3c) \geq f(c, -2c, c),$$
i.e.
$$9\lambda c + 3c(9\lambda^2 - 24\lambda)^{\frac{1}{2}} \geq 12\lambda c$$
or
$$9\lambda^2 - 24\lambda \geq \lambda^2,$$
since
$$\lambda \geq 3.$$

Therefore the hexagon belongs to type I and the result follows from theorem 1.

(*b*) Let $\frac{8}{3} \leq \lambda < 3$. Then the condition (4·1) is easily seen to be satisfied. The equations (4·2) which define a and b are clearly equivalent to (15·1).

It will now be shown that the region belongs to type II. For this it will suffice to prove that along A_1A_2, $f(x,y,z)$ is a decreasing function of x (see preliminary remarks). The equation of A_1A_2 is
$$a^2+ab+b^2 = ax - by = (a+b)x + bz.$$
Therefore, along A_1A_2,

$$\frac{df}{dx} = 3\lambda\left(1+\frac{dz}{dx}\right) + \tfrac{1}{2}\{9\lambda^2(x+z)^2 - 24\lambda(z-x)^2\}^{-\frac{1}{2}}\left\{18\lambda^2(x+z)\left(1+\frac{dz}{dx}\right) - 48\lambda(z-x)\left(\frac{dz}{dx}-1\right)\right\}$$
$$= \frac{1}{b}[9\lambda^2(x+z)^2 - 24\lambda(z-x)^2]^{-\frac{1}{2}}[24\lambda(z-x)(a+2b) - 9\lambda^2(x+z)$$
$$\times a - 3a\lambda\{9\lambda^2(x+z)^2 - 24\lambda(z-x)^2\}^{\frac{1}{2}}]. \tag{15·4}$$

As both A_1 and $D_1(b-a, -a-2b, 2a+b)$ lie on the first sector boundary, the value of $3\lambda(x+z) + \{9\lambda^2(x+z)^2 - 24\lambda(z-x)^2\}^{\frac{1}{2}}$ at both A_1 and D_1 is the same, so that at the point A_1
$$3\lambda(x+z) + \{9\lambda^2(x+z)^2 - 24\lambda(z-x)^2\}^{\frac{1}{2}} > 3\lambda(a+2b).$$

Therefore, at A_1 along A_1A_2,

$$\frac{df}{dx} < K[8(b-a)(a+2b) - 3a\lambda(a+2b)], \tag{15·5}$$

where K is a positive constant. As $\lambda \geqslant \frac{8}{3}$ and $b \leqslant 2a$, it follows from (15·5) that at A_1, along A_1A_2, $\frac{df}{dx} < 0$, so that the region belongs to type II.

To complete the proof of the theorem it has only to be shown that the lattice $\mathscr{L}_1$ is admissible. For this lemma 17 is applied with $D = 6$.

The points of $\mathscr{L}_1$ with ξ, η, given by

$$0 < \xi^2 + \xi\eta + \eta^2 \leqslant 6,$$

are no other than $\pm A_1$, $\pm A_2$, $\pm A_3$, $\pm 2A_1$, $\pm 2A_2$, $\pm 2A_3$, $\pm C_1$, $\pm C_2$, and $\pm C_3$. They are all known to lie outside R. Therefore it has only to be proved that

$$f\{0, -\surd 7(a^2+ab+b^2), \quad \surd 7(a^2+ab+b^2)\} \geqslant 12\lambda c,$$

i.e. $$3\lambda\{\surd 7(a^2+ab+b^2)\} + \{63\lambda^2(a+ab+b^2) - 168\lambda(a^2+ab+b^2)\}^{\frac{1}{2}} \geqslant 12\lambda c.$$

By lemma 1·1 $$a^2+ab+b^2 \geqslant 3c^2.$$

Therefore $$3\lambda\{\surd 7(a^2+ab+b^2)\} \geqslant 3\lambda c\surd 21 > 12\lambda c,$$

and the theorem is completely proved.

16. Hyperbolic hexagons

Before taking up the hyperbolic regions it will be convenient to prove the following particular case of lemma 17.

Lemma 19. *Let*

$$f\{0, -\surd 19(a^2+ab+b^2), \surd 19(a^2+ab+b^2)\} \geqslant f(c, -2c, c).$$

Then $\mathscr{L}_1$ is admissible if the points A_1+2A_2 and A_1+3A_2 do not lie inside R.

Proof. Because of the hypothesis it follows from lemma 17 that $\mathscr{L}_1$ is admissible if all $\xi A_1 + \eta A_2$ with $0 < \xi^2+\xi\eta+\eta^2 \leqslant 18$ lie outside R. It is easy to see that the only integers less than 19 which can be expressed as $\xi^2+\xi\eta+\eta^2$ are 1, 3, 4, 7, 9, 12, 13 and 16. The corresponding points $\xi A_1 + \eta A_2$ can be obtained by rotations through multiples of $\frac{1}{3}\pi$ from the points A_1, A_1+A_2, $2A_1$, A_1+2A_2, $2A_1+A_2$, $3A_1$, $4A_1$, A_1+3A_2, $2A_1+2A_2$ and $3A_1+A_2$. Therefore the lemma will follow if all these points lie outside R. For this it has only to be shown that $2A_1+A_2$ and $3A_1+A_2$ do not lie inside R. This is an easy consequence of the convexity condition, since, for all $n \geqslant 1$, nA_1 and $n(A_1+A_2)$ lie in the part of the first sector external to R.

Theorem 7. *Let R be a region with hexagonal symmetry bounded by hyperbolic arcs. Let $(c, -2c, c)$ be the mid-point of one of these arcs and let the asymptotic angles of the hyperbolas be $180° - 2\theta$. Write*

$$m_1 = \frac{1}{\surd 3}\tan\theta; \text{ then}$$

(*a*) *If $0 < m_1 \leqslant \frac{\surd 5}{3}$, $\Delta(R) = \Delta$ and the only critical lattice is generated by the mid-points of the boundary arcs.*

(*b*) *If $\frac{\surd 5}{3} < m_1 \leqslant 1$, then $\Delta(R) \geqslant \Delta$, where*

$$\Delta = \frac{2}{\surd 3}(a^2+ab+b^2), \tag{16·1}$$

and $$4c^2 = (a+b)^2 - m_1^2(a-b)^2 = (a+2b)^2 - 9a^2m_1^2. \tag{16·2}$$

However, if m_1^2* $\leqslant \frac{53}{57} = 0{\cdot}929\ldots$, $\Delta(R)$ *is equal to* Δ *and there are just two critical lattices.*

Proof. The perpendicular distance of (x, y, z) from the line $x = z$ is easily seen to be $\frac{1}{\sqrt{3}}(x-z)$. Therefore the equation of the boundary in the first sector is given by

$$y^2 - m_1^2(x-z)^2 = 4c^2,$$

i.e.

$$f(x, y, z) = \{y^2 - m_1^2(x-z)^2\}^{\frac{1}{2}} = 2c. \tag{16·3}$$

Now the two parts of the theorem will be proved separately.

(*a*) Since $0 < m_1 \leqslant \frac{\sqrt{5}}{3}$,

$$f(0, -3c, 3c) = \{9c^2(1-m_1^2)\}^{\frac{1}{2}} \geqslant 2c.$$

Therefore the region belongs to type I and the result follows on application of theorem 1.

(*b*) As $m_1 > \frac{\sqrt{5}}{3}$, the region belongs to types II, III or IV. Equations (4·2) of lemma 1 which define a and b are easily seen to be equivalent to (16·2). The relation (16·2), on dividing by b^2 and writing $a/b = j$, gives

$$8m_1^2 j^2 + (2m_1^2 - 2)j - 3 - m_1^2 = 0. \tag{16·4}$$

It will now be proved that R belongs to types II or III. As explained in the preliminary remarks, it will suffice to prove that

$$a^2 + ab + b^2 \leqslant (a+2b)\,c. \tag{16·5}$$

Write

$$\begin{aligned} F &= 4(a^2+ab+b^2)^2 - 4(a+2b)^2 c^2 \\ &= 4(a^2+ab+b^2)^2 - (a+2b)^2\{(a+b)^2 - m_1^2(a-b)^2\} \\ &= b^4[j^4(3+m_1^2) + j^3(2+2m_1^2) - j^2(1+3m_1^2) - j(4+4m_1^2) + 4m_1^2] \\ &= b^4(j-1)\,[j^3(3+m_1^2) + j^2(5+3m_1^2) + 4j - 4m_1^2] \\ &= b^4(j-1)\,\phi(j). \end{aligned} \tag{16·6}$$

Substituting for $8m_1^2 j^2$ and $64m_1^4 j^3$ from (16·4), one gets

$$\begin{aligned} 64m_1^4\phi(j) &= 64m_1^4(4j - 4m_1^2) \\ &\quad + (5+3m_1^2)\,8m_1^2[(3+m_1^2) - (2m_1^2-2)j] \\ &\quad + (3+m_1^2)\,[8m_1^2(3+m_1^2)j + (2-2m_1^2)\{(3+m_1^2) + (2-2m_1^2)j\}] \\ &= j(12 + 132m_1^2 + 276m_1^4 - 36m_1^6) + (18 + 114m_1^2 + 102m_1^4 - 234m_1^6) \\ &= \psi(j). \end{aligned} \tag{16·7}$$

As $m_1^2 \leqslant 1$, both the terms in $\psi(j)$ are non-negative, so that $\psi(j) \geqslant 0$. Further $j \leqslant 1$, therefore it easily follows from (16·7) and (16·6) that (16·5) is true and R belongs to types II or III, to both of which theorem II applies.

Now, in order to complete the proof, it has only to be shown that, for $m_1^2 \leqslant \frac{53}{57}$, the lattice $\mathscr{L}_1$ is admissible. By lemma 19 this will follow if one can prove that

(i) $A_1 + 2A_2$ lies outside R,

(ii) $A_1 + 3A_2$ lies outside R, and

(iii) $f\{0, -\sqrt{19(a^2+ab+b^2)}, \sqrt{19(a^2+ab+b^2)}\} \geqslant 2c$.

Proof of (ii). From table 1 the co-ordinates of $A_1 + 3A_2$ are $(4a+3b, -a-4b, b-3a)$. As both $-a-4b$ and $b-3a$ are negative, $A_1 + 3A_2$ lies in the second sector. The boundary of R in the second sector has the equation

$$x^2 - m_1^2(z-y)^2 = 4c^2 = (a+2b)^2 - 9a^2m_1^2.$$

* Δ (R) is probably equal to Δ for other values of m_1 too, but I have not been able to prove it.

Therefore, it has to be shown that

$$(4a+3b)^2-m_1^2(5b-2a)^2 \geqslant (a+2b)^2-9a^2m_1^2,$$

i.e.

$$15a^2+20ab+5b^2+5a^2m_1^2+20abm_1^2-25b^2m_1^2 \geqslant 0. \tag{16·8}$$

Since $a \geqslant \frac{1}{2}b$ and $m_1^2 \leqslant 1$, it follows that

$$15a^2+(20ab-10b^2m_1^2)+(5b^2-5b^2m_1^2)+5a^2m_1^2+(20abm_1^2-10b^2m_1^2) \geqslant 0,$$

so that (16·8) and hence (ii) is proved.

Proof of (*i*). The co-ordinates of A_1+2A_2 are $(3a+2b, -a-3b, b-2a)$. As both $-a-3b$ and $b-2a$ are non-positive, A_1+2A_2 lies in the second sector. Therefore, one has to prove that

$$(3a+2b)^2-m_1^2(4b-a)^2 \geqslant (a+2b)^2-9a^2m_1^2,$$

i.e.

$$8a^2+8ab+8a^2m_1^2+8abm_1^2-16b^2m_1^2 \geqslant 0,$$

or

$$\phi(j) = j^2(1+m_1^2)+j(1+m_1^2)-2m_1^2 \geqslant 0. \tag{16·9}$$

Substituting for $8m_1^2j^2$ from (16·4), one has

$$\begin{aligned} 8m_1^2\phi(j) &= 8m_1^2(1+m_1^2)j-16m_1^4+(1+m_1^2)\{(3+m_1^2)-(2m_1^2-2)j\} \\ &= j(1+m_1^2)(6m_1^2+2)+3+4m_1^2-15m_1^4 \\ &\geqslant \tfrac{1}{2}(1+m_1^2)(6m_1^2+2)+3+4m_1^2-15m_1^4 \\ &= 4+8m_1^2-12m_1^4 \\ &\geqslant 0, \end{aligned}$$

since $j \geqslant \frac{1}{2}$ and $m_1^2 \leqslant 1$. From this (16·9) and hence (i) follows.

Proof of (*iii*).

$$f\{0, -\sqrt{19(a^2+ab+b^2)}, \sqrt{19(a^2+ab+b^2)}\} = \{19(a^2+ab+b^2)(1-m_1^2)\}^{\frac{1}{2}}.$$

Therefore (iii) is equivalent to

$$19(a^2+ab+b^2)(1-m_1^2) \geqslant 4c^2.$$

As $a^2+ab+b^2 \geqslant 3c^2$, and $m_1^2 \leqslant \frac{53}{57}$,

$$19(a^2+ab+b^2)(1-m_1^2)-4c^2 \geqslant c^2(53-57m_1^2) \geqslant 0.$$

This completes the proof.

By more detailed arguments it is possible to prove (iii) for $m_1 \leqslant 0·9317\ldots$, and hence to prove that $\Delta(R) = \Delta$ for $\frac{5}{9} < m_1^2 \leqslant 0·9317$.

17. Twelve-sided stars

Theorem 8. *Let R be a star-shaped equal-sided dodecagon with centre at the origin, remoter vertices on the lines $x=0$, $y=0$ and $z=0$, and nearer vertices on the lines $y=z$, $z=x$ and $x=y$. Let the co-ordinates of one of the vertices be $(c, -2c, c)$ and the vertical angle be $2\theta = 2\tan^{-1}m$.*

(*a*) *If $60° \leqslant 2\theta \leqslant 120°$, then $\Delta(R) = 2c^2\sqrt{3}$, and the only critical lattice is generated by the nearer vertices.*

(*b*) *Let $x_1 = 0·422\ldots$ be a root of the equation*

$$3x^4-2x^3-36x^2-6x+9 = 0. \tag{17·1}$$

Suppose that $x_1 \leqslant m\sqrt{3} = 3m_1 < 1$, then $\Delta(R) = \Delta$, where

$$\Delta = \frac{2}{\sqrt{3}}(a^2+ab+b^2), \tag{17·2}$$

and

$$c(1+3m_1) = a+(2b+a)m_1 = (b-a)+(3a+3b)m_1. \tag{17·3}$$

Also, there are just two critical lattices.

Proof. As $m_1 = \frac{1}{\sqrt{3}} m$, the equation of the boundary lines is easily seen to be

$$f(x,y,z) = c(1+3m_1),$$

where

$$f(x,y,z) = \begin{cases} x+m_1(z-y), & \text{when } 0 \leqslant x \leqslant z, \\ z+m_1(x-y), & \text{when } 0 \leqslant z \leqslant x, \\ -z+m_1(x-y), & \text{when } 0 \leqslant -z \leqslant -y, \text{ and} \\ -y+m_1(x-z), & \text{when } 0 \leqslant -y \leqslant -z. \end{cases} \tag{17·4}$$

For $2\theta \leqslant 120°$, R is easily seen to be a region with hexagonal symmetry. Now take up parts (*a*) and (*b*) separately.

(*a*) As $2\tan^{-1} m = 2\theta \geqslant 60°$, therefore $m \geqslant \frac{1}{\sqrt{3}}$ and $m_1 \geqslant \frac{1}{3}$. Therefore

$$f(0, -3c, 3c) - f(c, -2c, c) = 6m_1 c - (1+3m_1)\, c \geqslant 0,$$

so that R is a region of type I and part (*a*) of the theorem follows on application of theorem 1.

(*b*) Now $0{\cdot}422\ldots \leqslant 3m_1 < 1$, and R is obviously a region of type IV. Equations (4·2) and (4·7) which define a, b and Δ are clearly equivalent to (17·2) and (17·3). From (17·3) one gets

$$a/b = (1+m_1)/2(1-m_1). \tag{17·5}$$

In order to apply theorem 3 it has first to be shown that the area condition postulated therein is satisfied. As explained in the preliminary remarks it will suffice to prove that if $P(x_1, y_1, z_1)$ is any point on the boundary between $A_1(a, -a-b, b)$ and $T_1(c, -2c, c)$, and if p, q, r are real numbers with $p+q+r = 0$ for which the points $(p \pm \frac{1}{2}x_1, q \pm \frac{1}{2}y_1, r \pm \frac{1}{2}z_1)$ lie on the second sector boundary, then

$$\Delta\sqrt{3} \leqslant |\, p(z_1-y_1) - x_1(r-q)\,| = |\, p(2z_1+x_1) - x_1(2r+p)\,|, \tag{17·6}$$

with the sign of equality only when P coincides with T_1 and $m_1 = m'$ (say).

Since $x+y+z = 0$ for all points in the plane, one obtains from (17·4) after eliminating the middle co-ordinates

$$x_1 + m_1(2z_1 + x_1) = c(1+3m_1), \tag{17·7}$$

$$-\tfrac{1}{2}z_1 - r + m_1(x_1 + 2p + \tfrac{1}{2}z_1 + r) = c(1+3m_1), \tag{17·8}$$

and

$$p - \tfrac{1}{2}x_1 + r - \tfrac{1}{2}z_1 + m_1(p - \tfrac{1}{2}x_1 - r + \tfrac{1}{2}z_1) = c(1+3m_1). \tag{17·9}$$

By addition and subtraction, (17·8) and (17·9) give

$$-\tfrac{1}{2}(x_1+2z_1)\,(1-m_1) + p(1+3m_1) = 2c(1+3m_1), \tag{17·10}$$

and

$$(p+2r)\,(1-m_1) - \tfrac{1}{2}x_1(1+3m_1) = 0. \tag{17·11}$$

Therefore

$$\begin{aligned}(1+3m_1)\,(1-m_1)\,\{p(2z_1+x_1) - x_1(2r+p)\} \\ = 2c(1-3m_1)\,(1-m_1)\,(x_1+2z_1) + \tfrac{1}{2}(x_1+2z_1)^2\,(1-m_1)^2 - \tfrac{1}{2}x_1^2(1+3m_1)^2,\end{aligned}$$

so that

$$\begin{aligned}(1+3m_1)\,(1-m_1)\,\{p(2z_1+x_1) - x_1(2r+p) - \Delta\sqrt{3}\} \\ = 2c(1+3m_1)\,(1-m_1)\,(x_1+2z_1) + \tfrac{1}{2}(x_1+2z_1)^2\,(1-m_1)^2 \\ - \tfrac{1}{2}x_1^2(1+3m_1)^2 - (1+3m_1)\,(1-m_1)\,\Delta\sqrt{3} \\ = \Phi(x_1, z_1) \quad \text{(say)}.\end{aligned} \tag{17·12}$$

Since $0<m_1<1$, the area condition will obviously be satisfied if it can be shown that at all points on A_1T_1, $\Phi(x_1, z_1)>0$, except that $\Phi=0$ at T_1 for $m_1=\overline{m}$. Along A_1T_1 from A_1 to T_1, x increases continuously. Therefore it will suffice to prove that

(i) Along A_1T_1, Φ is a strictly increasing function of x, and

(ii) At the point T_1, $\Phi>0$, except if $m_1=\overline{m}$, when $\Phi=0$.

Now, the equation of A_1T_1 is

$$x+m_1(2z+x)=c(1+3m_1).$$

Therefore along A_1T_1

$$\frac{dz}{dx}=-\frac{1+m_1}{2m_1},$$

and
$$\frac{d\Phi}{dx}=-x(1+3m_1)^2+\{2c(1+3m_1)(1-m_1)+(x_1+2z_1)(1-m_1)^2\}\left\{1-\frac{1+m_1}{m_1}\right\}<0.$$

This proves (i). Now to prove (ii).

Since $b=2a(1-m_1)/(1+m_1)$, (17·3) gives

$$c(1+3m_1)=a\left\{(1+m_1)+\frac{4m_1(1-m_1)}{1+m_1}\right\}=\frac{a}{(1+m_1)}(1+6m_1-3m_1^2), \tag{17·13}$$

and
$$\Delta\surd3=2(a^2+ab+b^2)$$
$$=2a^2\left\{1+\frac{2(1-m_1)}{1+m_1}+\frac{4(1-m_1)^2}{(1+m_1)^2}\right\}$$
$$=\frac{2a^2}{(1+m_1)^2}(7-6m_1+3m_1^2). \tag{17·14}$$

From (17·12), (17·13) and (17·14) it follows that at $T_1(c, -2c, c)$

$$\Phi=\tfrac{1}{2}\{12(1+3m_1)(1-m_1)c^2+9(1-m_1)^2c^2-(1+3m_1)^2c^2-2(1+3m_1)(1-m_1)\Delta\surd3\}$$
$$=\frac{a^2}{2}\left[\frac{(20-36m_1^2)(1+6m_1-3m_1^2)^2}{(1+m_1)^2(1+3m_1)^2}-\frac{4(1+3m_1)(1-m_1)(7-6m_1+3m_1^2)}{(1+m_1)^2}\right]$$
$$=\frac{4a^2(3m_1-1)}{(1+m_1)^2(1+3m_1)^2}(27m_1^4-6m_1^3-36m_1^2-2m_1+1)$$
$$=\frac{4a^2(3m_1-1)}{(1+m_1)^2(1+3m_1)^2}\Psi(m_1)\quad\text{(say)}. \tag{17·15}$$

By the rule of signs, $\Psi(m_1)=0$ cannot have more than two positive roots. Further

$$\Psi(0)>0,\quad \Psi(\tfrac{1}{3})<0\quad\text{and}\quad\Psi(\infty)>0.$$

Therefore $\Psi(m_1)=0$ has exactly two positive roots $\overline{m}$ and m' and

$$0<\overline{m}<\tfrac{1}{3},\ \tfrac{1}{3}<m'<\infty. \tag{17·16}$$

Consequently
$$\Psi(m_1)=(m_1-\overline{m})(m_1-m')\overline{\Psi}(m_1),$$

where $\overline{\Psi}(m_1)$ is positive for all positive m_1. From the expression for $\Psi(m_1)$ in (17·15) it is now clear that $\overline{m}$ is the root in the interval $(0, \frac{1}{3})$ of

$$27m_1^4-6m_1^3-36m_1^2-2m_1+1=0.$$

Write $3\overline{m}=x_1$. Then x_1 is the root in $(0, 1)$ of

$$3x^4-2x^3-36x^2-6x+9=0.$$

Now for all $x_1=3\overline{m}<3m_1<1$, $\Psi(m_1)<0$. Also $\Psi(\overline{m})=0$. Therefore it follows from (17·14) that at T_1, $\Phi>0$ for all $x_1=3\overline{m}<3m_1<1$ and $\Phi=0$ for $m_1=\overline{m}$. This proves (ii), so that the area condition of theorem 3 is satisfied.

In view of all this the theorem will follow by the application of theorem 3 if it can be shown that (*a*) $\mathscr{L}_1$ and $\mathscr{L}_2$ are admissible, and (*b*) for $m_1 = \overline{m}$, the lattice generated by $(c, -2c, c)$ and $(2c, -c, -c)$ is not admissible.

By the corollary to lemma 18, (*b*) follows from the

LEMMA. *Let p, q and r be defined by* (17·10) *and* (17·11) *with both x_1 and z_1 replaced by c. Then the point $(p-\frac{3}{2}c, q+3c, r-\frac{3}{2}c)$ does not lie on the boundary of R.*

Proof. Write

$$x' = p-\frac{3c}{2} = 2c+\frac{3c(1-m_1)}{2(1+3m_1)}-\frac{3c}{2} = \frac{2c}{(1+3m_1)} \quad \text{(by 17·10)}$$

and

$$y' = -p-r+3c = -\tfrac{1}{2}\left\{\frac{c(1+3m_1)}{2(1-m_1)}+\frac{c(7+9m_1)}{2(1+3m_1)}\right\}+3c$$

$$= \frac{-2c(1+m_1)}{(1-m_1)(1+3m_1)}+3c$$

$$= \frac{c(1+4m_1-9m_1^2)}{(1-m_1)(1+3m_1)}.$$

Since $m_1 < \frac{1}{3}$, it is easily seen that $0 \leqslant y' \leqslant x'$. The equation of the boundary of R for $0 \leqslant y \leqslant x$ is

$$y+m_1(2x+y) = c(1+3m_1).$$

Now

$$y'+m_1(2x'+y') = c\left\{\frac{(1+m_1)(1+4m_1-9m_1^2)}{(1-m_1)(1+3m_1)}+\frac{4m_1}{(1+3m_1)}\right\}$$

$$= \frac{c(1+9m_1-9m_1^2-9m_1^3)}{(1-m_1)(1+3m_1)}.$$

Therefore, if (x', y') lay on the boundary of R, it would follow that

$$1+9m_1-9m_1^2-9m_1^3 = (1-m_1)(1+3m_1)^2 = 1+5m_1+3m_1^2-9m_1^3,$$

or

$$4m_1(1-3m_1) = 0,$$

which is not true, since m_1 is neither zero nor $\frac{1}{3}$. Therefore the lemma and hence also (*b*) follows.

To prove (*a*) apply lemma 17 with $D = 6$. For $0 < \xi^2+\xi\eta+\eta^2 \leqslant 6$, the points of $\mathscr{L}_1$ are easily seen to lie on the boundary of R. Therefore, it has only to be proved that

$$f\{0, -\surd 7(a^2+ab+b^2), \surd 7(a^2+ab+b^2)\} \geqslant c(1+3m_1),$$

i.e.

$$2m_1\surd 7(a^2+ab+b^2) \geqslant c(1+3m_1).$$

Using (17·13) and (17·14), this is equivalent to

$$0 \leqslant 28(a^2+ab+b^2)\,m_1^2-(1+3m_1)^2c^2$$

$$= \frac{28m_1^2(7-6m_1+3m_1^2)\,a^2}{(1+m_1)^2}-\frac{a^2(1+6m_1-3m_1^2)^2}{(1+m_1)^2}$$

$$= \frac{a^2}{(1+m_1)^2}(75m_1^4-132m_1^3+166m_1^2-12m_1-1)$$

$$= \frac{a^2}{(1+m_1)^2}\chi(m_1) \quad \text{(say)}. \tag{17·18}$$

Now
$$\frac{dx(m_1)}{dm_1} = 300m_1^3 - 396m_1^2 + 332m_1 - 12$$
$$= 300m_1^3 + 132m_1(1-3m_1) + 200m_1 - 12$$
$$> \tfrac{1}{3}.200\,(0{\cdot}4) - 12$$
$$> 0,$$

since $0{\cdot}4 < 3m_1 < 1$. Therefore, for $x_1 = 0{\cdot}422\ldots \leqslant 3m_1 < 1$,

$$x(m_1) > x\left(\frac{0{\cdot}4}{3}\right)$$
$$= \tfrac{1}{27}\{25(0{\cdot}4)^4 - 132(0{\cdot}4)^3 + 498(0{\cdot}4)^2 - 108(0{\cdot}4) - 27\}$$
$$> 0.$$

Thus (17·18) and consequently (a) is proved. This completes the proof of the theorem.

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